# What a metric can do for you*

## Geometric approach to Quantum Poincaré recurrence

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## Classical Mechanics- Symplectic Geometry

The phase space $M$ of classical mechanics $(q,p)$ can be understood as a symplectic $2N$-dimensional manifold $(M,\omega)$ endowed with a closed, non-degenerate 2−form

$$\omega = \sum_{a=1}^{N} dp_a \wedge dq^a \, .$$

On $M$ one can define a smooth real-valued function $H : M \to \mathbb{R}$, known as the *Hamiltonian* or the energy function. The Hamiltonian induces a special vector field on the symplectic manifold $X_H : M \to TM$, known as the *Hamiltonian vector field*

$$dH(Y) = \omega(Y, X_H) \, ,$$

for any $Y$ in $\chi(M)$.

## Liouville's theorem

The Hamiltonian vector field $H$ induces a *Hamiltonian flow* $\varphi_t : \mathbb{R} \times M \to M$ on the manifold. The parameter of this one-parameter family of transformations of the manifold is commonly called the time. The Liouville's theorem states that the Hamiltonian flow preserves the volume form on the phase space:

$$\int_{\varphi_t(\Omega)} \omega^N = \int_{\Omega} \omega^N \, ,$$

for any precompact domain $\Omega \in M$ (being $\omega^N = p_1 \wedge q^1 \wedge \cdots \wedge p_N \wedge q^N$).

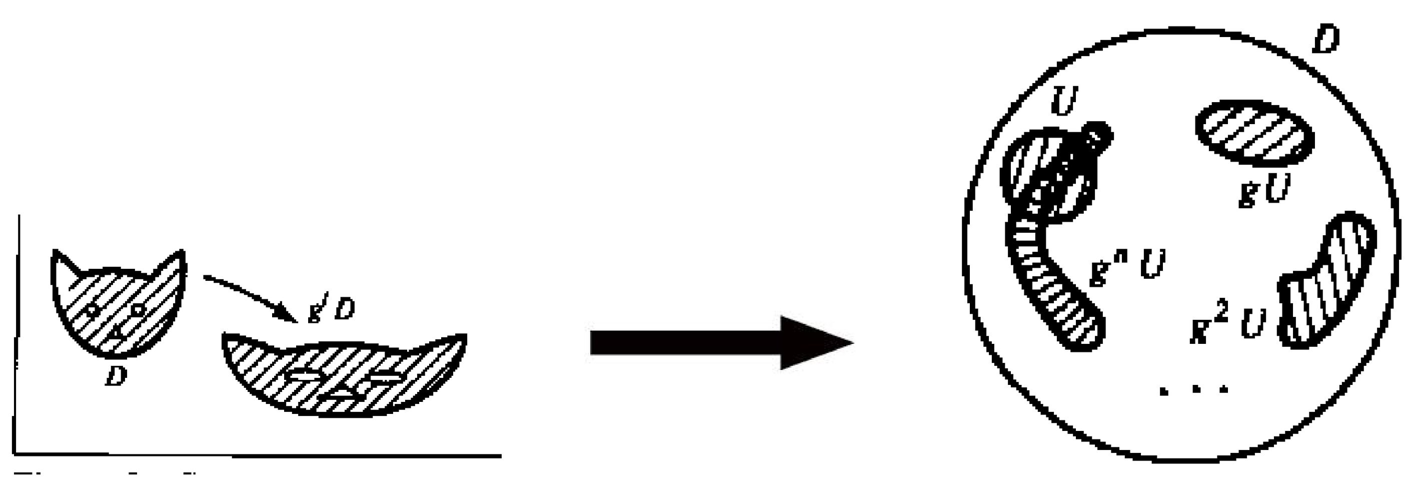
## Poincaré recurrence Theorem

If $M$ is compact, $(M,\omega^N)$ is a finite measure space, and then $\varphi_t : M \to M$ is a measure-preserving transformation. Hence, for any $\Omega$ with $\int_\Omega \omega^N \neq 0$, the set of points $x$ of $\Omega$ such that $\varphi_t^k(x) \notin \Omega$ for all $k > 0$ has zero measure.

## Graphically

Images from Arnold's book "Mathematical methods of classical mechanics"

**Liouville** $\Longrightarrow$ **Poincaré**




Since the Hamiltonian flow preserves the volume form, then for any domain, recurrence appears.

## Quantum mixed states - Principal fibre bundle

A quantum mixed state can be described by a density matrix. A density matrix is a complex matrix $\rho$ that satisfies the following properties:

1. $\rho = \rho^\dagger$.
2. $\rho \geqslant 0$.
3. $\mathrm{tr}(\rho) = 1$.

The set of strictly positive (or faithful) density matrices

$$\mathcal{P}^+ = \{\rho \in \mathcal{P} \,|\, \rho > 0\} \, ,$$

is the base manifold of the Uhlman's principal fibre bundle $\mathcal{S}(\mathcal{P}^+, U(n), \pi)$, being

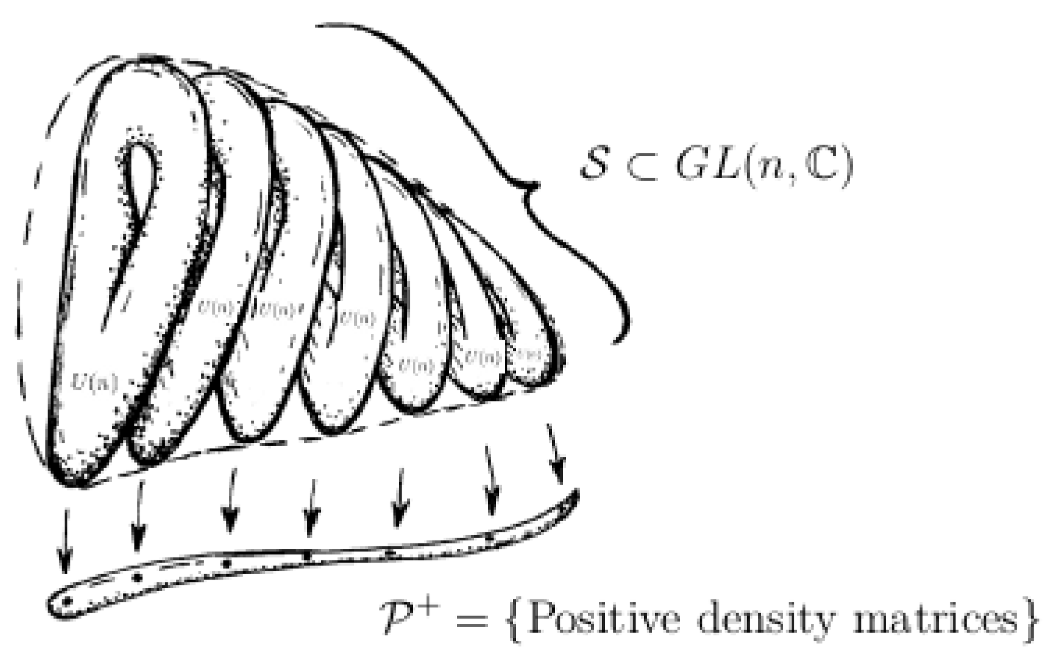
$$\mathcal{S} := \left\{ W \in \mathrm{GL}(n,\mathbb{C}) \, ; \, \mathrm{tr}(WW^\dagger) = 1 \right\} , .$$

the principal space, the structure group, the unitary group $U(n)$ of dimension $n$, and the projection to the base $\pi : \mathcal{S} \to \mathcal{P}^+$ given by

$$\pi(W) = WW^\dagger \quad .$$

## Uhlmann's principal fibre bundle

Modified picture from Roger Penrose's book "The road to reality"




## Hamiltonian vector field and dynamic metric

A Hamiltonian operator $H$ induces a vector field $h$ in the principal space $h : \mathcal{S} \to T\mathcal{S}$ given by

$$h(W) := -iHW.$$

And a Riemannian metric $g_H$ on $T_W\mathcal{S}$ given by

## Dynamic Riemannian metric

$$g_H(X,Y) := \frac{1}{2}\mathrm{tr}(X^\dagger H^{-2} Y + Y^\dagger H^{-2} X).$$

## Quantum mechanics - Riemannian geometry

### Theorem (Main theorem)

*We can state and prove that:*

1. *$h$ is a Killing vector field of $(\mathcal{S}, g_H)$.*
2. *The integral curves $\gamma : I \subset \mathbb{R} \to \mathcal{S}$ of $h$ are geodesics of $(\mathcal{S}, g_H)$.*
3. *The projection on the base manifold $\mathcal{P}^+$ of the geodesic $\gamma$ satisfies the von Neumann equation*

$$\frac{d}{dt}\pi \circ \gamma = -i\,[H, \pi \circ \gamma] \, .$$

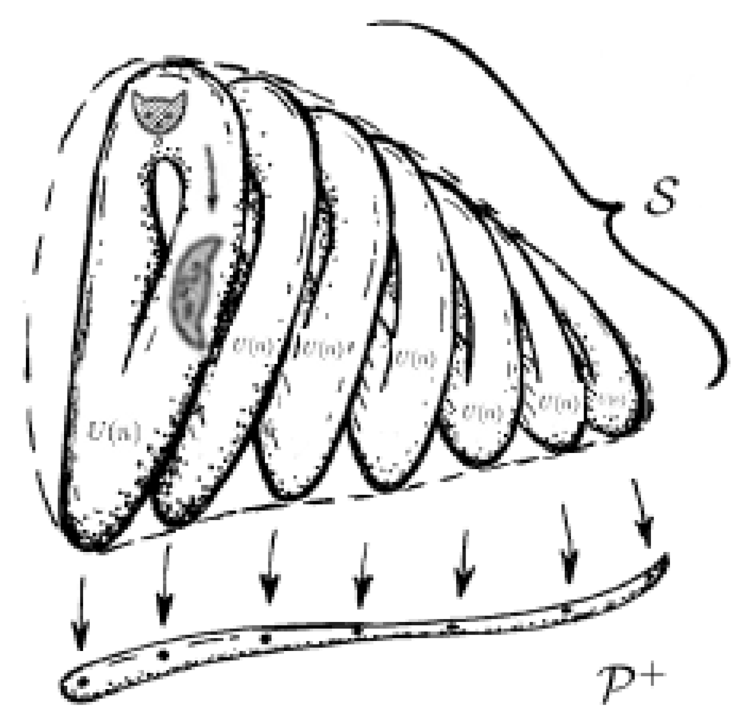
Moreover, every conserved observable ( $A = A^\dagger, \quad [H,A] = 0$ ) induces a Killing vector field $\mathcal{A}$ in $(\mathcal{S}, g_H)$ given by

$$\mathcal{A}(W) = iAW.$$

**Physical symmetries $\Longrightarrow$ geometric symmetries.**

## Corollary - Liouville type theorem

Since $h$ is a Killing vector field, its integral flow preserves the Riemannian volume




## Quantum Poincaré recurrence theorem

From the Liouville principle we can deduce in analogous way to the classical one, the Poincaré recurrence theorem for quantum mixed states.

## More information

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* Title inspired in the title of a chapter of the book "The road to reality" written by Roger Penrose

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