Compactifications of reductive groups, non-abelian symplectic cutting and geometric quantisation of non-compact spaces

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joint work with Michael Thaddeus (Columbia University)

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Let G be (connected) split reductive group over a field (i.e. over $\mathbb{C}$, $G = K_{\mathbb{C}}$, with K compact Lie group)

e.g. G =semi-simple, $GL(n, \mathbb{C}), (\mathbb{C}^*)^n, \text{Spin}^c_{\mathbb{C}}, \ldots$

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Question

What are 'good' compactifications $\overline{G}$ of G?

Here 'good' should mean

- $G \times G$-equivariant
- smooth, with all orbit closures smooth
- boundary $\overline{G} \setminus G$ is a *smooth normal crossing divisor*
- nice enumeration of orbits

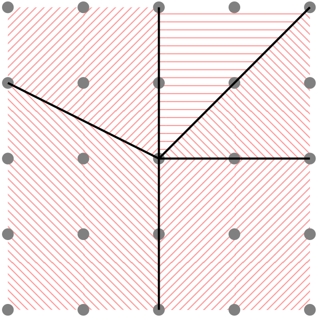
Ideally want some conceptual understanding of boundary $\overline{G} \setminus G$

modular compactification?

Toric varieties & fans

Toric varieties $\overline{T}$ are normal T-equivariant varieties with open dense orbit

Determined by fans: collection of strongly convex, rational cones in $\Lambda_T \otimes_{\mathbb{Z}} \mathbb{Q}$



- every cone simplicial $\Rightarrow$ at worst finite quotient singularities
- non-minimal element on ray $\Rightarrow$ extra orbifold-structure
- fan complete $\Rightarrow$ $\overline{T}$ compact

Wonderful compactfication of adjoint groups

G adjoint, i.e. $ZG = \{1\}$
e.g. $PGL(n)$, $SO(2n+1, \mathbb{C})$, E_8, F_4, G_2

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λ regular dominant weight
highest weight representation V_λ

have

$$\begin{array}{ccc} G & \hookrightarrow & End(V_\lambda) \\ & \searrow & \downarrow \\ & & \mathbb{P}(End(V_\lambda)) \end{array}$$

Wonderful compactification

Definition (De Concini - Procesi)

The wonderful compactification $\overline{G}^{w}$ of an adjoint group is the closure in $\mathbb{P}(\mathrm{End}(V_\lambda))$

Independent of choice of λ

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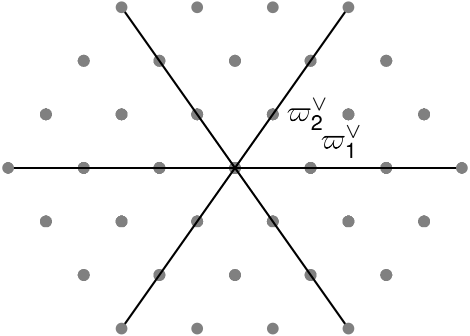
Independent of choice of λ

T_G maximal torus in G, take closure in $\overline{G}^w$
$\Rightarrow$ get toric variety $\overline{T_G}$

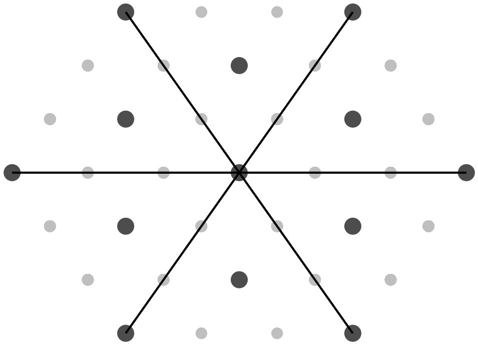
Fan of $\overline{T_G}$ = Weyl chambers + Λ_G

(Λ_G= co-weight lattice since G is adjoint) e.g. $PGL(3)$:



Smooth since $\varpi_i^\vee$ generate co-weight lattice!

e.g. $SL(3, \mathbb{C})$



Corresponding toric variety no longer smooth!

Moduli Problem

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Too many objects: stack is not separated nor of finite type

Cure this by imposing a stability condition

Which bundles are stable?

Theorem (Birkhoff-Grothendieck-Harder)

Every G-principal bundle on $\mathbb{P}^1$ reduces to the maximal torus and up to isomorphims is entirely determined by a co-character

$$\Lambda \ni \rho : \mathbb{G}_m \to G$$

unique up to W_G-action

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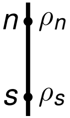
Take two charts given by stereographic projection from s and n, use ρ as transition function

Every $\mathbb{G}_m$- equivariant G-principal bundle on $\mathbb{P}^1$ is determined by action of $\mathbb{G}_m$ on fibers over n and s,



given by co-characters ρ_n and ρ_s, unique up to W_G.

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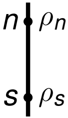

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Underlying non-equivariant bundle determined by $\rho_n - \rho_s$

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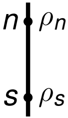

given by co-characters ρ_n and ρ_s, unique up to W_G.

Underlying non-equivariant bundle determined by $\rho_n - \rho_s$

Theorem (M.-Thaddeus)

Every $\mathbb{G}_m$-equivariant G-principal bundle on a chain-of-lines of length n reduces to the maximal torus T_G and is given up to isomorphism by an element of Λ^{n+1}/W_G

Σ-stable bundles

Choose a (stacky) fan Σ for T_G, satisfying:

- Σ is simplicial
- Σ is Weyl-invariant
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Choose ordering of integral elements

$$\rho_1, \ldots, \rho_j$$

on rays in positive Weyl chamber

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A bundle on chain of lines is Σ-*stable* if

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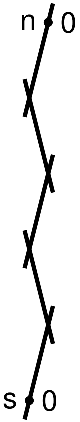
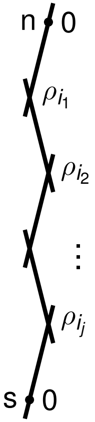


Definition

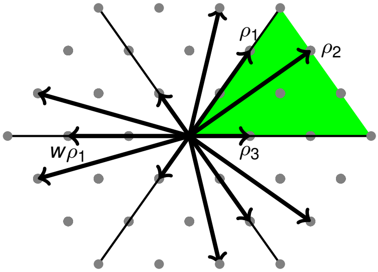
A bundle on chain of lines is *Σ-stable* if

- co-chars on on extremal *n* and *s* are 0
- co-chars on nodes are $\rho_{i_1}, \ldots, \rho_{i_j}$:
  in order and all rays of single cone of Σ



Example

e.g. $PGL(3)$



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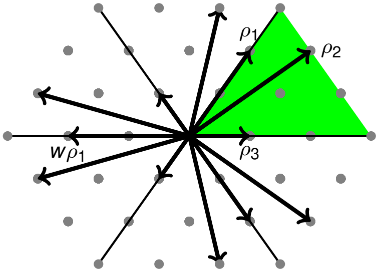

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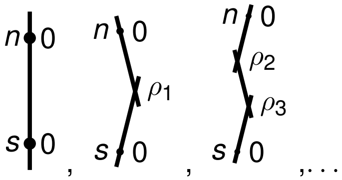

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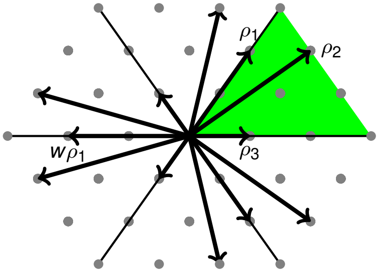

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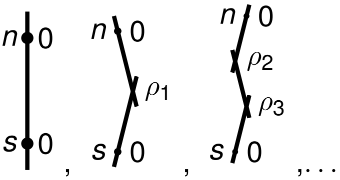
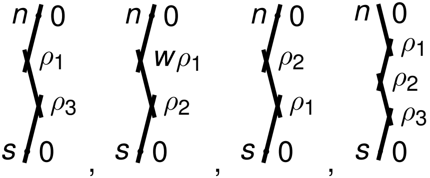
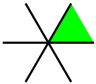


non-Σ-stable:



- G semi-simple:



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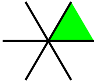

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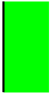

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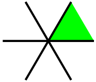

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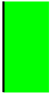

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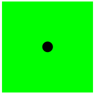
- $G = T$ torus



Weyl-chamber everything $\Rightarrow$ *any* fan refines Weyl-chamber

Σ stacky fan, rays generate Λ over $\mathbb{Q}$
have

$$1 \to L \to (\mathbb{G}_m)^N \to T_{\mathbb{C}} \to 1$$

Theorem (Cox)

$$\mathcal{M}_T(\Sigma) \cong (\mathbb{A}^N)^0 / L$$

If rays don't generate Λ, still have

$$\mathcal{M}_T(\Sigma) \cong \left((\mathbb{A}^N)^0 \times T_{\mathbb{C}}\right) / (\mathbb{G}_m)^N$$

Want to generalize this to arbitrary groups

We use *Vinberg monoid* (Vinberg, Rittatore, Alexeev-Brion)
Given G reductive, have

$$\begin{array}{c} S_G \\ \downarrow \\ \mathbb{A}_\Pi \end{array}$$

affine reductive semigroup, units: $(G \times T)/ZG \subset S_G$

$\mathbb{A}_\Pi$ is smooth affine toric variety,
fan=pos Weyl chamber in co-weight lattice

Property

$$S_G /\!\!/ T \cong \overline{G_{\mathrm{ad}}}$$

Using the data of the fan, can now base-change Vinberg monoid:

$$\begin{array}{ccc} \mathbb{A}^N \times_{\mathbb{A}^\Pi} S_G & \longrightarrow & S_G \\ \downarrow & & \downarrow \\ \mathbb{A}^N & \longrightarrow & \mathbb{A}_\Pi \end{array}$$

Theorem (M.-Thaddeus)

$$\mathcal{M}(\Sigma) \cong \left(\mathbb{A}^N \times_{\mathbb{A}^\Pi} S_g\right)^0 / (\mathbb{G}_m)^N$$

global quotient by torus, if Σ dual to P GIT quotient or symplectic reduction
If $\mathcal{M}_{T_G}(W\Sigma)$ semi-projective, so is $\mathcal{M}_G(\Sigma)$

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Related to symplectic cut:
T compact, $T \circlearrowright M$, $\mu : M \to \mathfrak{t}^*$, P polytope $\subset \mathfrak{t}^*$

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Abelian symplectic cutting (Lerman)

$$M_P := \mu^{-1}(P)/\sim \qquad \mu(M_P) = \mu(M) \cap P$$

$X(P)$ toric manifold determined by P
Can re-interpret Delzant construction as symplectic cut of T^*T:

Master-cut

$$X(P) = T^*T_P, \qquad M_P = \left(M \times T^*T_P\right) /\!\!/ T$$

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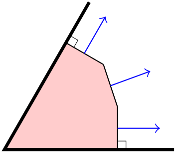
Problem for geometric quantisation:

Property (Woodward)

Even if M is Kahler, M_P need not be!

How to understand this as a symplectic reduction / global quotient?

Apply this cut to $K \circlearrowright K_{\mathbb{C}} \cong T^*K$. If all normals in $\mathfrak{t}_+$



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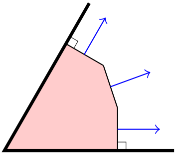

Theorem

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Can use this to construct all other cuts as global (Kahler!) quotients:

Non-abelian master cut

For general M Hamiltonian K-space have

$$M_P \cong (M \times (T^*K)_P) /\!\!/ K.$$

For non-compact M with proper moment maps, Weitsman (2001) and Paradan (2009) use cutting to construct quantizations

Formal geometric quantization

$$Q_K^{-\infty}(M) = \lim_{n\to\infty} Q_K(M_{nP})$$

Our work gives local surgery description of this construction