The Stochastics of Energy Markets

...or...

Modelling Financial Energy Forwards

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Overview

- **Goal:** Model the forward price dynamics in power markets
- **Why?**
  - Price and hedge options and other derivatives
  - Risk management (hedge production and price risk)

1. Some stylized facts of energy forward prices
2. Levy processes in Hilbert space
   - Subordination of Wiener processes
3. Modelling the forward dynamics
   - Adopting the Heath-Jarrow-Morton (HJM) dynamical modelling from interest rate theory
4. Ambit fields and forward prices
   - A direct HJM approach
1. Forward markets
Energy forward contracts

- Forward contract: a promise to deliver a commodity at a specific *future* time in return of an agreed price
  - Examples: coffee, gold, oil, orange juice, corn....
  - or.... temperature, rain, electricity
- Electricity: future delivery of power over a period in time
  - A given week, month, quarter or year
- The agreed price is called the *forward price*
  - Denominated in Euro per MWh
  - Forward contracts traded at EEX, NordPool, etc...
  - Financial delivery!
• Forward price at time $t \leq T_1$, for contract delivering over $[T_1, T_2]$, denoted by $F(t, T_1, T_2)$

• Connection to fixed-delivery forwards

$$F(t, T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} f(t, T) \, dT$$

• Musiela parametrization: $x = T_1 - t, y = T_2 - T_1$

$$G(t, x, y) = F(t, t + x, t + x + y), \quad g(t, x) = f(t, t + x)$$

• Focus on modelling the dynamics of the forward curve

$$t \mapsto g(t, x)$$
Some stylized facts of power forwards

- Consider the *logreturns* from observed forward prices (at NordPool)

\[ r_i(t) = \ln \frac{F(t, T_{1i}, T_{2i})}{F(t - 1, T_{1i}, T_{2i})} \]

- General findings are:
  1. Distinct heavy tails across all segments
  2. No significant skewness
  3. Volatilities (stdev's) are in general falling with time to delivery
     \[ x = T_1 - t \] (Samuelson effect)
  4. Significant correlation between different maturities \( x \)
     (idiosyncratic risk)
• Fitting NIG and normal to logreturns of forwards by maximum likelihood
- Expected logreturn (left) and volatility (right)
• Plot of log-correlation as a function of years between delivery
• Correlation decreases in general with distance between delivery
  • ...but in a highly complex way
Summary of empirical evidence

- Forward curve $g(t, x)$ is a random field in time and space
  - Or, a stochastic process with values in a function space
- Strong dependencies between maturity times $x$
  - High degree of idiosyncratic risk in the market
- Non-Gaussian distributed log-returns
  - Dynamics is not driven by Brownian motion
2. Hilbert space-valued Lévy processes
• Goal: construct a Hilbert-space valued Lévy process with given characteristics
  • For example, a normal inverse Gaussian (NIG) Lévy process in Hilbert space
• $X$ is a $d$-dimensional NIG random variable if

$$X | \sigma^2 \sim \mathcal{N}_d(\mu + \beta \sigma^2, \sigma^2 C)$$

• $\mu \in \mathbb{R}^d$, $\beta \in \mathbb{R}$, $C$ $d \times d$ covariance matrix,
• $\sigma$ an inverse Gaussian random variable
• $X$ defined by a mean-variance mixture model
Lévy processes by subordination

- Define a NIG Lévy process $L(t)$ with values in Hilbert space by subordination
- In general: let
  - $H$ be a separable Hilbert space
  - $\Theta$ a real-valued subordinator, that is, a Lévy process with increasing paths
  - $W$ a drifted $H$-valued Brownian motion with covariance operator $Q$ and drift $b$
  - $Q$ is symmetric, positive definite, trace-class operator,

\[
\text{Cov}(W)(f,g) = \mathbb{E} [\langle W(1) - b, f \rangle \langle W(1) - b, g \rangle ] = \langle Qf, g \rangle
\]

- Define

\[
L(t) = W(\Theta(t))
\]
• Let $\psi_{\Theta}$ be the cumulant (log-characteristic) function of $\Theta$

• Cumulant of $L$ becomes

$$
\psi_L(z) = \psi_{\Theta}\left(i\langle z, b \rangle - \frac{1}{2}\langle Qz, z \rangle \right), \ z \in H
$$

• Let $(a, 0, \ell)$ be characteristic triplet of $\Theta$, then triplet of $L$ is $(\beta, aQ, \nu)$

$$
\beta = ab + \int_0^{\infty} \mathbb{E}[1(|W(t)| \leq 1)] \ell(dz)
$$

$$
\nu(A) = \int_0^{\infty} P^{W(t)}(A) \ell(dt), \ A \subset H, \text{ Borel}
$$
• Suppose $L$ square-integrable Lévy process
• Define covariance operator

$$\text{Cov}(L)(f, g) = \mathbb{E} [\langle L(1), f \rangle \langle L(1), g \rangle] = \langle Qf, g \rangle$$

• Supposing mean-zero Lévy process
• $Q$ symmetric, positive definite, trace-class operator

• If $L$ is defined via subordination, covariance operator is

$$Q = \mathbb{E}[\Theta(1)]Q$$

• Supposing $\Theta(1)$ integrable
• So, how to obtain $L$ being NIG Lévy process?

• Choose $\Theta$ to be driftless inverse Gaussian Lévy process, with Lévy measure

$$\ell(dz) = \frac{\gamma}{2\pi z^3} e^{-\delta^2 z^2/2} 1(z > 0) \, dz$$

• Define $L(t) = W(\Theta(t))$, which we call a $H$-valued NIG Lévy process with triplet $(\beta, 0, \nu)$,

**Theorem**

$L$ is a $H$-valued NIG Lévy process if and only if $TL(t)$ is a $\mathbb{R}^n$-valued NIG Lévy process for every linear operator $T : H \mapsto \mathbb{R}^n$. 
3. Forward price dynamics
• Let $H$ be a separable Hilbert space of real-valued continuous functions on $\mathbb{R}_+$
  • with $\delta_x$, the evaluation map, being continuous
  • $x \in \mathbb{R}_+$ is time-to-maturity
  • $H$ is, e.g. the space of all absolutely continuous functions with derivative being square integrable with respect to an exponentially increasing function (Filipovic 2001)

• Assume $L$ is square-integrable zero-mean Lévy process
  • Defined on a separable Hilbert space $U$, typically being a function space as well (e.g. $U = H$)
  • Triplet $(\beta, Q, \nu)$ and covariance operator $Q$
• Define process \( X \) on \( H \) as the solution of

\[
dX(t) = (AX(t) + a(t)) \, dt + \sigma(t) \, dL(t)
\]

• \( A = d/dx \), generator of the \( C_0 \)-semigroup of shift operators on \( H \)
• \( a(\cdot) \) \( H \)-valued process, \( \sigma(\cdot) \) \( L_{HS}(\mathcal{H}, H) \)-valued process being predictable
  • \( L_{HS}(\mathcal{H}, H) \), space of Hilbert-Schmidt operators, \( \mathcal{H} = Q^{1/2}(U) \)

\[
\mathbb{E} \left[ \int_0^t \| \sigma(s) Q^{1/2} \|_{L_{HS}(U, H)}^2 \, ds \right] < \infty
\]

• \( \sigma \) and \( a \) may be functions on the state again
  • We will not assume that generality here
• Mild solution, with $S$ as shift operator

$$X(t) = S(t)X_0 + \int_0^t S(t - s)a(s) \, ds + \int_0^t S(t - s)\sigma(s) \, dL(s)$$

• Define forward price $g(t, x)$ by

$$g(t, x) = \exp(\delta_x(X(t)))$$

• By letting $x = T - t$, we reach the actual forward price dynamics

$$f(t, T) = g(t, T - t)$$
• Assume $X$ is modelled under "risk-neutrality", then $f(\cdot, T)$ must be a martingale
  • Yields conditions on $a$ and $\sigma$!

• Introduce

\[
\hat{a}(t) = \int_0^t a(s)(T - s) \, ds, \quad \hat{\sigma}(t) = \int_0^t \delta_0 S(T - s)\sigma(s) \, dL(s)
\]

**Theorem**

*The process* $t \mapsto f(t, T)$ *for* $t \leq T$ *is a martingale if and only if*

\[
d\hat{a}(t) = -\frac{1}{2} d[\hat{\sigma}, \hat{\sigma}]^c(t) - \{e^{\Delta \hat{\sigma}(t)} - 1 - \Delta \hat{\sigma}(t) - \}
\]

• $\Delta \hat{\sigma}(t) = \hat{\sigma}(t) - \hat{\sigma}(t-)$, $[\hat{\sigma}, \hat{\sigma}]^c$ continuous part of bracket process of $\hat{\sigma}$
Market dynamics

- Forward model under risk neutral probability $\mathbb{Q}$
- Esscher transform $\mathbb{Q}$ to "market probability" $\mathbb{P}$ to get market dynamics of $F$
- Let $\phi(\theta)$ be the log-moment generating function (MGF) of $L$
  - Recall characteristic triplet of $L$ as $(\beta, Q, \nu)$
  - Assume $L$ is exponentially integrable

\[
\phi(\theta) = \ln \mathbb{E}[e^{(\theta, L(1))U}] \\
= (\beta, \theta)U + \frac{1}{2}(Q\theta, \theta)U \\
+ \int_U e^{(\theta, y)U} - 1 - (\theta, y)U 1_{|y|U \leq 1} \nu(dy), \theta \in U
\]
• $d\mathbb{P}/d\mathbb{Q}$ conditioned on $\mathcal{F}_t$ has density

$$Z(t) = \exp((\theta, L(t))_{\mathbb{U}} - \phi(\theta) t)$$

• Lévy property of $L$ preserved under Esscher transform

• Characteristic triplet under $\mathbb{P}$ is $(\beta_\theta, Q, \nu_\theta)$

$$\beta_\theta = \beta + \int_{|y| \leq 1} y \nu_\theta(dy), \quad \nu_\theta(dy) = e^{(\theta, y)}_{\mathbb{U}} \nu(dy)$$

• $\theta \in \mathbb{U}$ is the market price of risk
  • Esscher transform will shift the drift in $X$-dynamics, and
  • and rescale (exponentially tilt) the jumps of $L$
Example

- \( L = W \), Wiener process in \( U \)
- Bracket process can be computed to be

\[
[\hat{\sigma}, \hat{\sigma}]^c(t) = \int_0^t \| \delta_0 S(T-s)\sigma(s)Q^{1/2} \|_{L_{HS}(U,\mathbb{R})}^2 ds
\]

- An example by Audet et al. (2004)
- Volatility specification
  - \( \sigma \) multiplication operator: \( \delta_x \sigma(t)u = \eta e^{-\alpha x} u(x), \ u \in U \)
  - \( \eta, \alpha \) positive constants, \( \alpha \) mean-reversion speed
  - Volatility structure linked to an exponential Ornstein-Uhlenbeck process for the spot
- **Spatial covariance structure of $W$**
  - Let $Q$ be integral operator
  - $q(x, y) = \exp(-\kappa|x - y|)$ integral kernel
- **Recall correlation structure from empirical studies...**
  - ...close to exponentially decaying
  - Some seasonal variations: let $\eta$ be seasonal
- **Forward dynamics of Audet et al. (2004)**

\[
\ln \frac{g(t, x)}{g(0, x)} = -\frac{1}{2}\eta^2 \int_0^t e^{-2\alpha(x+t-s)} \, ds + \int_0^t \eta e^{-\alpha(x+t-s)} \, dW(s, x)
\]

- Or....

\[
\frac{df(t, T)}{f(t, T)} = \eta e^{-\alpha(T-t)} \, dW(t, T - t)
\]
• Note: series representation of $W$
  • Independent Gaussian processes, $\{e_n\}$ basis of $U$

$$W(t) = \sum_{n=1}^{\infty} \langle W(t), e_n \rangle U e_n$$

• May represent the dynamics in terms of Brownian factors
  • Infinite factor model
• Recall the heavy tails in log-return data for NordPool forwards
  • A Wiener specification $W$ is not justified
• Should use an exponential NIG-Lévy dynamics instead
  • Choose $L$ to be NIG, constructed by subordinator
  • Keep covariance operator
Numerical examples with NIG-Levy field

- Simulation of forward field by numerically solving the hyperbolic stochastic partial differential equation for $X$
  - Euler discretization in time
  - A finite-element method in "space" $x$
  - Conditions at "inflow" boundary $x = \infty$ and at $t = 0$
- Initial condition $X(0, x)$ is "today’s observed forward curve" on log-scale
  - Exponentially decaying curve
  - Motivated from "typical" market shapes
- Boundary condition at infinity equal to constant
  - Stationary spot price dynamics yield a constant forward price at "infinite maturity"
• $L$ is supposed to be a NIG-Łévy process, which is defined as a subordination

• Appeal to the series expansion of $W$, which is truncated in the numerics
  • Simulate a path of an inverse Gaussian Łévy process
  • Change time of the finite set of independent Brownian motions
  • Sum up these scaled by eigenvalues and basis function to get the NIG-Łévy field approximation

• Parameters
  • $\alpha = 0.2$, mean-reversion
  • $\kappa = 2$, correlation
  • IG-parameters chosen by convenience ($\gamma = 10, \delta = 1$)
• Forward field, for \( x = 0, \ldots, 40 \) days to maturity, and \( t \) daily over 4 years. Implied spot process for \( x = 0 \)

• Can we recover the spot dynamics from the forward model?
Implied spot price dynamics

• One can recover the spot dynamics as

\[ g(t, 0) = \exp(\delta_0(X(t))) \]

• \( X \) is driven by by NIG Lévy process in \( U \)
  • ”Infinitely” many Lévy processes
• For \( \tilde{L} \) is univariate NIG Lévy process, \( \tilde{\sigma} \) stochastic process on \( \mathbb{R} \), it holds

\[ \delta_0 \int_0^t \sigma(s) \, d\tilde{L}(s) = \int_0^t \tilde{\sigma}(s) \, d\tilde{L}(s) \]

• Spot can be represented as a dynamics in terms of a univariate NIG Lévy process
4. HJM modeling by ambit fields
Forward dynamics by ambit fields

- A twist on the HJM approach
  - by direct modelling rather than as the solution of some dynamic equation
  - Barndorff-Nielsen, B., Veraart (2010b)

- Simple arithmetic model in the risk-neutral setting

\[
g(t, x) = \int_{-\infty}^{t} \int_{0}^{\infty} k(t - s, x, y)\sigma(s, y)L(dy, ds)
\]

- \( L \) is a Lévy basis, \( k \) non-negative deterministic function, 
  \( k(u, x, y) = 0 \) for \( u < 0 \), stochastic volatility process \( \sigma \) 
  (typically independent of \( L \) and stationary)
• **$L$ is a Lévy basis on $\mathbb{R}^d$** if
  1. the law of $L(A)$ is infinitely divisible for all bounded sets $A$
  2. if $A \cap B = \emptyset$, then $L(A)$ and $L(B)$ are independent
  3. if $A_1, A_2, \ldots$ are disjoint bounded sets, then

  $$L(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} L(A_i), \text{ a.s}$$

• **Stochastic integration in time and space:** use the Walsh-definition (for *square integrable* Lévy bases)
  - Natural adaptedness condition on $\sigma$
  - square integrability on $k(t - \cdot, x, \cdot) \times \sigma$ with respect to covariance operator of $L$

• **Possible to relate ambit fields to Hilbert-space valued processes**
Martingale condition

- No-arbitrage conditions: \( t \mapsto f(t, T) := g(t, T_t) \) must be a martingale

**Theorem**

\( f(t, T) \) is a martingale if and only if there exists \( \tilde{k} \) such that

\[
k(t - s, T - t, y) = \tilde{k}(s, T, y)
\]

- Note, cancellation effect on \( t \) in 1st and 2nd argument ensures martingale property
Example 1: exponential damping function (motivated by OU spot models)

\[ k(u, x, y) = \exp(-\alpha(u + x + y)) \]

Satisfies the martingale condition

\[ k(t - s, T - t, y) = \exp(-\alpha(y + T - s)) =: \tilde{k}(s, T, y) \]

Example 2: the SPDE specification of \( f \)

- Let \( L = W \), a univariate Brownian motion for simplicity

\[ dg(t, x) = \frac{\partial g}{\partial x}(t, x) \, dt + \sigma(t, x) \, dW(t) \]
• Solution of the SPDE

\[ g(t, x) = g_0(x + t) + \int_0^t \sigma(s, x + (t - s)) \, dW(s) \]

• Note: forward price \( g(t, x) \) is an ambit process
• Letting \( x = T - t \),

\[ g(t, T - t) = g_0(T) + \int_0^t \sigma(s, T - s) \, dW(s) \]

• Martingale condition is satisfied....of course!
Example

• Suppose $k$ is a weighted sum of two exponentials
  • Motivated by a study of spot prices on the German EEX
  • ARMA(2,1) in continuous time

$$k(t - s, x, y) = w \exp(-\alpha_1(t - s + x + y)) + (1 - w) \exp(-\alpha_2(t - s + x - y))$$

• $L = \mathcal{W}$ a Gaussian basis
• $\sigma(s, y)$ again an ambit field
  • Exponential kernel function
  • Driven by inverse Gaussian Lévy basis
• Spot is very volatile
• Rapid convergence to zero when time to maturity increases
  • In reality there will be a seasonal level
Thank you for your attention!
References

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