Graph-theoretic methods in combinatorial (algebraic) topology

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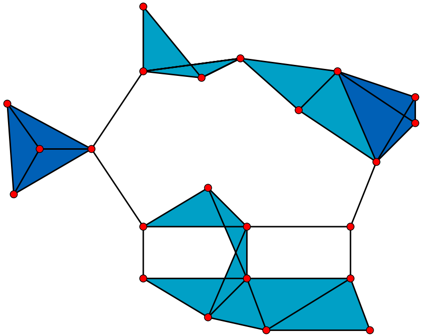
Combinatorial (algebraic) topology

- complexes arising from combinatorial objects,
- applications of topology to combinatorics,
- computational aspects,
- triangulations, face numbers,
- embeddability,
- applied algebraic topology,
- probabilistic topology.

Flag complexes

- If G is a graph, then the clique complex $\mathrm{Cl}(G)$ is the simplicial complex whose faces are the cliques (complete subgraphs) of G.



Source: Wikipedia

- a.k.a. flag complexes,
- Vietoris-Rips complexes,
- order complexes $\Delta\mathcal{P}$ of posets,
- simplicial curvature a la Gromov.

Complexity of $H_*(K)$

Problem (Kaibel, Pfetsch, *Algorithmic Problems in Polytope Theory*)

Given a simplicial complex K, presented by the list of maximal faces, what is the complexity of calculating $H_*(K)$?

$$K$$

$$\Downarrow$$

$$\cdots \leftarrow C_{n-1}(K) \leftarrow C_n(K) \leftarrow C_{n+1}(K) \leftarrow \cdots$$

$$\Downarrow$$

$$H_n(K)$$

The first stage seems to require exponential time.

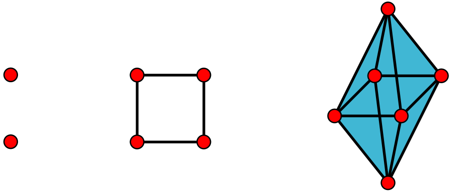
NP-hardness

NP-hard problems = decision (Yes/No) problems which provably require more than polynomial time (unless $P = NP$)

- Take an instance $\mathcal{I}$ of your favorite problem $\mathcal{P}$ which you already know is NP-hard.
- Construct a simplicial complex $K = K(\mathcal{I})$ and $n \in \mathbb{N}$ such that
$$H_n(K) = 0 \iff \mathcal{I} \text{ is a Yes-instance}$$
- The homology problem is then "at least as hard" as $\mathcal{P}$.

Hyperoctahedral spheres

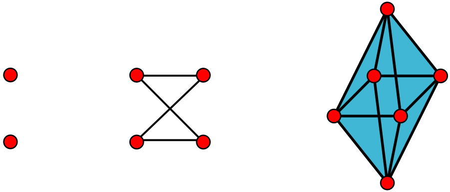
O_0 $\quad$ O_1 $\quad$ O_2 $\quad$ O_3



ΣO_2

These are the clique complexes of the graphs:

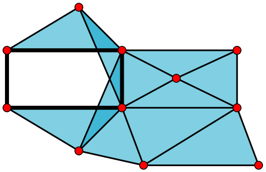
K_2 $\quad$ $K_{2,2}$ $\quad$ $K_{2,2,2}$ $\quad$ $K_{2,2,2,2}$



...

$O_n \hookrightarrow K$	$\Rightarrow$	$\alpha \in H_n(K)$
some face of O_n is a *maximal* face of K	$\Rightarrow$	$\alpha \neq 0$
$K_{\underbrace{2,2,\ldots,2}_{n+1}} \hookrightarrow G$	$\Rightarrow$	$\alpha \in H_n(\mathrm{Cl}(G))$
some clique in $K_{\underbrace{2,2,\ldots,2}_{n+1}}$ is a maximal clique of G	$\Rightarrow$	$\alpha \neq 0$

"*n*-gadget in *G*"



$n = 1$

The main result

Theorem (MA+JS)

There is a class of graphs (cochordal), such that

- *For every graph G in the class every group $H_n(\mathrm{Cl}(G))$ is generated by n-gadgets.*
- *Given a graph G in the class and an integer n it is NP-hard to decide if G contains an n-gadget.*

Consequences for complexity of $H_*(-)$

Theorem

The following problems are NP-hard

- *Given a graph G and an integer n, decide if $H_n(\mathrm{Cl}(G)) = 0$ (remains NP-hard even restricted to cochordal graphs).*
- *Given any simplicial complex K, presented as the list of maximal faces, and an integer n, decide if $H_n(K) = 0$.*

Let $K =$ Alexander dual of $\mathrm{Cl}(G)$.

- $H_n(K) = H^{|G|-n-3}(\mathrm{Cl}(G))$,
- max-faces of K are the complements of non-edges of G.

Flag spheres

Problem (Hopf)

The Euler characteristic of a $2n$-dimensional manifold M of non-positive sectional curvature satisfies $(-1)^n\chi(M) \geq 0$.

Charney and Davis (1995) develop a local, combinatorial analogue.

Charney-Davis conjecture

Problem (Charney, Davis)

If K is a $(2s-1)$-dimensional flag sphere then

$$\sum_i (-\frac{1}{2})^i f_i(K) \geq 0.$$

- The C.-D. conjecture implies the Hopf conjecture for manifolds with a cubical cell decomposition.
- Equality holds for the hyperoctahedral spheres, their various subdivisions and...

Theorem (Davis, Okun (2001))

If K is a flag 3-sphere with f_0 vertices and f_1 edges then

$$f_1 \geq 5f_0 - 16.$$

Face numbers of flag spheres

Combinatorial characterization of f-vectors of flag d-spheres.

$d = 1$	obvious
$d = 2$	obvious
$d = 3$	known up to possibly a finite number of cases, [Davis-Okun, Gal, MA+JH]
$d = 4$	known, [Davis-Okun, Gal, Nevo-Murai]
$d = 2s - 1 \geq 5$	one non-trivial restriction $$f_1 \leq \frac{s-1}{2s} f_0^2 + f_0$$ for sufficiently large f_0 [MA]

Upper bound for f_1 in flag 3-spheres

Thm: If G – graph with n vertices, m edges and $K = \mathrm{Cl}(G) = S^3$ then $m \leq \frac{1}{4}n^2 + n$.

- $\mathrm{lk}_K v = S^2$,
- $\mathrm{lk}_K v$ does not contain the subgraph $K_{3,3}$,
- G does not contain the subgraph $K_{1,3,3}$,
- (Erdös) for large n, the maximizer of $|E(G)|$ among $K_{1,3,3}$-free graphs is

  fig

- for this graph $m = \frac{1}{4}n^2 + n$ and $\mathrm{Cl}(G) = S^1 * S^1 = S^3$.

In the general case use van Kampen – Flores:

$$\mathrm{Cl}(K_{\underbrace{3,\ldots,3}_{n+1}}) \not\hookrightarrow S^{2n}.$$

Background

Theorem (Mantel, Turan)

If G is a graph with n vertices, m edges and no triangles then

$$m \leq \frac{1}{4}n^2$$

and the maximizer is $K_{n/2,n/2}$.

$K = \mathrm{Cl}(G) = S^3$, $f_0 = n, f_1 = m, f_2 = 2(m-n) \approx n^2$

Theorem (*Stability*, Erdös, Simonovitz, Lovasz)

If G is a graph with n vertices, $m \geq \frac{1}{4}n^2$ edges and only $\approx n^2$ triangles then G is "very similar" to $K_{n/2,n/2}$.

The stability method — a general approach

- Suppose G is very dense ($m \geq \frac{1}{4}n^2$) and $\mathrm{Cl}(G) = S^3$
- Stability $\Rightarrow$ G is similar to $K_{n/2,n/2}$
- G has extra geometric properties which can be used to show that in fact G must look like

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Results

Theorem (MA+JH, conjectured by Gal)

If K is a flag 3-sphere with the number of edges close to maximum, precisely

$$\frac{1}{4}f_0^2+\frac{1}{2}f_0+17\le f_1\le\frac{1}{4}f_0^2+f_0$$

and f_0 is sufficiently large, then K is still a join of two cycles.

Theorem (MA)

The inequality

$$f_1\le\frac{s-1}{2s}f_0^2+f_0$$

holds for a large class of $(2s-1)$-dimensional weak pseudomanifolds with sufficiently many vertices, including in particular $(2s-1)$-dimensional spheres, homology spheres, closed manifolds, homology manifolds and more.