Always-cartesian cubes and wrong-way maps

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- Further details may be found on the arxiv: 1304.1662v1 [math.AT]
- To appear in the Arolla Conference Proceedings.

Goal:

To describe a classification of square diagrams of spaces

$$\begin{array}{ccc} \mathscr{X}_{\emptyset} & \longrightarrow & \mathscr{X}_{0} \\ \downarrow & & \downarrow \\ \mathscr{X}_{1} & \longrightarrow & \mathscr{X}_{01} \end{array}$$

which . . .

- . . . are homotopy pullbacks (reps. push outs)
- . . . retain this property under equivalence-preserving functors.

I will explain

- homotopy pullback/pushout diagrams (and generalizations)
- where this classification problem arose
- a conjecture for higher dimensional cubes.

Warm-up: “Cobar” Construction

As topologists, we study “Spaces”. $\Rightarrow$ replace by something more tractable, e.g. a **resolution** of some sort, (oft (co)simplicial); Example: **cobar construction** of comm rings, $B \to A$

$$A \rightrightarrows A \otimes_B A \rightrightarrows\!\!\!\!\to \cdots$$

really, $A \otimes_B -$ applied to

$$\mathrm{sk}_0\, \Delta^\bullet := (\ [0] \rightrightarrows [1] \rightrightarrows\!\!\!\!\to [2] \cdots\ )$$

Topologically, apply $X * -$ to same object, get

$$CX \rightrightarrows \Sigma X \rightrightarrows\!\!\!\!\to \Sigma X \vee \Sigma X \cdots$$

Warmup ctd.

Such (cosimplicial) objects can be approximated by their finite truncations,...

$$X * \mathrm{sk}_0\, \Delta^\bullet = (CX \rightrightarrows \Sigma X \;\substack{\longrightarrow\\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \Sigma X \vee \Sigma X \cdots)$$

...but, re-assembled into spaces.

This “re-assembly” (for simp'l/cosimp'l) is a (hocolim/holim).

Proposition

These finite ho(co)lims can be calculated using cubes.

$$\begin{aligned} \mathrm{holim}(tr_1(X * \mathrm{sk}_0\, \Delta^\bullet)) &\simeq \mathrm{holim}(CX \to \Sigma X \leftarrow CX) \\ &:= Map\left(\begin{array}{ccc} & & \Delta^0 \\ & & \downarrow \\ \Delta^0 & \longrightarrow & \Delta^1, \end{array} \quad \begin{array}{ccc} & & CX \\ & & \downarrow \\ CX & \longrightarrow & \Sigma X \end{array} \right) \end{aligned}$$

Great! *Literature: Blakers-Massey, Loday, Goodwillie (Calc2 §2), Munson-Volic, ...*

Definitions!

- $[n] = \{0, 1, \dots, n\}$,
- $\mathscr{P}([n]) :=$ power set on $[n]$, morphs = inclusions.
- $\mathscr{P}_0([n]) := \mathscr{P}([n]) - \emptyset$, $\mathscr{P}^1([n]) := \mathscr{P}([n]) - [n]$
- An $(n+1)$-cube of spaces is a functor $\mathscr{X} : \mathscr{P}([n]) \to \text{Top}$;
- A square is a functor $\mathscr{P}([1]) \to \text{Top}$:

$$\begin{array}{ccc} \mathscr{X}_\emptyset & \longrightarrow & \mathscr{X}_0 \\ \downarrow & & \downarrow \\ \mathscr{X}_1 & \longrightarrow & \mathscr{X}_{01} \end{array} \qquad \begin{array}{ccc} \Omega X & \longrightarrow & PX \simeq \bullet \\ \downarrow & & \downarrow \\ \bullet \simeq PX & \longrightarrow & X \end{array} \qquad \begin{array}{ccc} X & \longrightarrow & CX \simeq \bullet \\ \downarrow & & \downarrow \\ \bullet \simeq CX & \longrightarrow & \Sigma X \end{array}$$

- $\mathscr{X}$ is a (ho)cartesian $(n+1)$-cube iff $\mathscr{X}_\emptyset \xrightarrow{\simeq} \operatorname{holim}_{\mathscr{P}_\emptyset([n])} \mathscr{X}$.
- $\mathscr{X}$ is a (ho)cocartesian $(n+1)$-cube iff $\mathscr{X}_{[n]} \xleftarrow{\simeq} \operatorname{hocolim}_{\mathscr{P}^1([n])} \mathscr{X}$.

Absolutely (co)cartesian

- F is a hofunctor if $X \simeq Y \Rightarrow F(X) \simeq F(Y)$.
- $\mathscr{X}$ is a cartesian $(n+1)$-cube iff $\mathscr{X}_\emptyset \simeq \operatorname{holim}_{\mathscr{P}_0([n])} \mathscr{X}$
- $\mathscr{X}$ is an **absolutely** cartesian $(n+1)$-cube iff $F(\mathscr{X})$ is cartesian $\forall$ hofunctors F; i.e. $F(\mathscr{X}_\emptyset) \simeq \operatorname{holim}_{\mathscr{P}_0([n])} F(\mathscr{X})$
- $\mathscr{X}$ is a cocartesian $(n+1)$-cube iff $\mathscr{X}_{[n]} \simeq \operatorname{hocolim}_{\mathscr{P}^1([n])} \mathscr{X}$.
- $\mathscr{X}$ is an **absolutely** cocartesian $(n+1)$-cube iff $F(\mathscr{X})$ is cocartesian $\forall$ hofunctors F; i.e. $F(\mathscr{X}_{[n]}) \simeq \operatorname{hocolim}_{\mathscr{P}^1([n])} F(\mathscr{X})$.
- Trivial example: constant square
- Nontrivial example: $\begin{array}{ccc} A & \longrightarrow & B \\ \| & & \| \\ A & \longrightarrow & B \end{array}$ (equivalences stable under pullback/pushout).

Classification

Theorem (E.)

A square of spaces $\mathscr{X}$,

$$\begin{array}{ccc} \mathscr{X}_\emptyset & \xrightarrow{h_1} & \mathscr{X}_0 \\ {\scriptstyle v_1}\downarrow & & \downarrow{\scriptstyle v_2} \\ \mathscr{X}_1 & \xrightarrow{h_1} & \mathscr{X}_{01} \end{array} \qquad\qquad \begin{array}{ccc} \mathscr{X}_1 & \longrightarrow & \mathscr{X}_0 \\ \| & & \| \\ \mathscr{X}_1 & \longrightarrow & \mathscr{X}_0 \end{array} \quad \begin{array}{ccc} \mathscr{X}_0 & = & \mathscr{X}_0 \\ \downarrow & & \downarrow \\ \mathscr{X}_1 & = & \mathscr{X}_1 \end{array}$$

is absolutely cartesian iff h_1, h_2 or v_1, v_2 are equivalences.

Remark

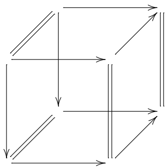
Holds for cocartesian if you allow contravariant functors.

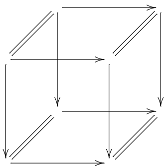
Conjectures

Conjecture (1)

$\mathscr{X} : \mathscr{P}([n]) \to$ Top absolutely cartesian iff $\exists Y, Z : \mathscr{P}([n-1]) \to$ Top absolutely cartesian such that $X : Y \to Z$.

e.g. dim 3, of the form



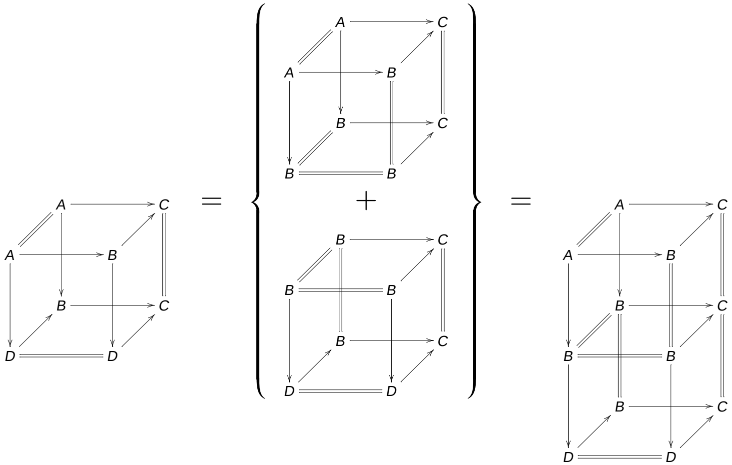


Conjecture (2)

$\mathscr{X} : \mathscr{P}([n]) \to$ Top absolutely cartesian iff absolutely cocartesian.

Counter of Conjecture (1)

Given maps $A \longrightarrow B \longrightarrow D \longrightarrow B \longrightarrow C$, can construct:



Conjecture 1 redux

Conjecture ((1)-False as stated)

$\mathscr{X} : \mathscr{P}([n]) \to$ *Top absolutely cartesian iff* $\exists Y, Z : \mathscr{P}([n-1]) \to$ *Top absolutely cartesian such that* $X : Y \to Z$.

Conjecture (1')

$\mathscr{X} : \mathscr{P}([n]) \to$ *Top absolutely cartesian iff "generated by" those cubes of the form in Conj (1). That is, a map between 2, a composition, or a combination of these operations.*

Conjecture 2

Conjecture (2)

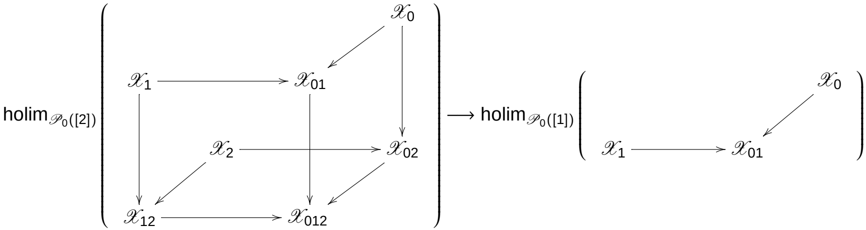
$\mathscr{X} : \mathscr{P}([n]) \to$ *Top absolutely cartesian iff absolutely cocartesian.*

Results:

- If assume spaces are simply connected and hofunctors land in simply connected spaces , then absolutely cartesian iff absolutely cartesian (proof method: clever use of $\Omega^\infty\Sigma^\infty$).
- If (absolutely cocartesian $\Rightarrow$ absolutely cartesian), then can show that (absolutely cartesian $\Rightarrow$ absolutely cocartesian (proof method: clever use of $\mathscr{X} \mapsto \mathrm{Map}(F(\mathrm{Map}(G(\mathscr{X}), A), B))$.

Applications/Wrong-Way maps

- FACT 1: Given map $A \to B$ of $(n+1)$-cubes with B cartesian : $\operatorname{holim}_{\mathscr{P}_0([n+1])}(A \to B) \simeq \operatorname{holim}_{\mathscr{P}_0([n])}(A)$
- FACT 2: Homotopy limits are contravariant with respect to maps of indexing categories; $I \to J \Rightarrow \operatorname{holim}_J \mathscr{X} \to \operatorname{holim}_I \mathscr{X}$. e.g. projection maps



Wrong- Way maps

$$\operatorname{holim}_{\mathscr{P}_0([2])}\left(\begin{array}{ccc} & & \mathscr{X}_0 \\ & & \swarrow \\ \mathscr{X}_1 & \longrightarrow & \mathscr{X}_{01} \end{array}\right)$$

$$\operatorname{holim}_{\mathscr{P}_0([2])}\left(\begin{array}{ccccc} & & & & \mathscr{Y}_0 \\ & & & \swarrow & \downarrow \\ \mathscr{Y}_1 & \longrightarrow & \mathscr{Y}_{01} & & \downarrow \\ \downarrow & & \downarrow & & \downarrow \\ \downarrow & \mathscr{Y}_2 & \longrightarrow & \longrightarrow & \mathscr{Y}_{02} \\ \downarrow & \swarrow & \downarrow & \swarrow & \\ \mathscr{Y}_{12} & \longrightarrow & \mathscr{Y}_{012} & & \end{array}\right)$$

Wrong- Way maps