Fibrancy of Symplectic Homology in Cotangent Bundles

Thomas Kragh

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- Let N be a closed smooth manifold, then T^*N has a canonical 1-form λ defined by

$$\lambda_{(q,p)}(v) = p(\pi_*(v)),$$

where $q \in N$, $p \in T_q^*N$, $v \in T_{(q,p)}(T^*N)$ and $\pi \colon\ T^*N \to N$.

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  where $q \in N$, $p \in T_q^*N$, $v \in T_{(q,p)}(T^*N)$ and $\pi\colon\ T^*N \to N$.
- Ex: (DT^*N, λ) is a Liouville domain - given any Riemannian structure on N.

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- If j is an exact Lagrangian embedding then the extension is exact.

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- Fukaya, Seidel and Smith's result was proven independently using slightly different techniques by Nadler.

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- Recall that the Hamiltonian flow is defined as the flow of X_H where X_H, solves $\omega(X_H, -) = dH$.

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- Then one defines the Floer homology $FH_*(H)$ as the Morse homology of A_H given by

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- For a a regular value we can restrict to $A_H^{-1}([a, \infty))$. We denote the resulting complex $FC_*^a(H)$.
- There is a natural quotient map

$$FH_*^a(H) \to FH_*^b(H) \qquad \text{for} \qquad a < b$$

Collar Neighborhood and action

- We may find a collar

$$X = (1-\varepsilon, 1] \times \partial M \subset M$$

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$$A_H(\gamma) = \int_\gamma \lambda - Hdt = \int_\gamma \lambda' - H(dt) = uh'(u) - h(u),$$

where $\{u\} \times \partial M$ is the leaf it lies in.

Symplectic Homology

- Now, fix a smooth map $f\colon (1-\varepsilon, \infty) \to \mathbb{R}$ such that
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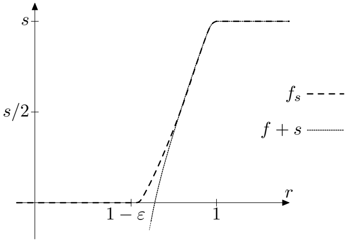

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  - f is concave,
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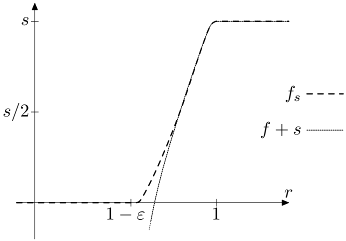
  - $f(t) \to -\infty$ when $t \to 1-\varepsilon$.
- Now define $f_s\colon \mathbb{R} \to \mathbb{R}$ as
  - $f_s(u) = f(u) + s$ for

    $f(u) > -s/2.$
  - $f_s(u) = 0$ for $s < 1-\varepsilon$.



  - All tangents to f_s intersecting the 2. axis above 0 is tangent at points where $f_s = f + s$.

Symplectic Homology

- We now define

$$H_s(z) = \begin{cases} f_s(u) & z = (u, x) \in X \\ 0 & z \in M - X \\ s & z \in T^*N - M \end{cases}$$

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$$SH_*(M) = \operatorname*{colim}_{s \to \infty} FH_*^{a}(H_s) \qquad \text{for any} \qquad a < 0.$$

Case of $M = DT^*L$

- When $M = DT^*L$ then the flow of these Hamiltonians reproduce geodesic flows on L and Viterbo calculated that

$$SH_*(DT^*L) \cong H_*(\mathcal{L}L)$$

when L is orientable and spinable.

Fiber-wise Symplectic Homology

- We now fix $q \in N$ and look at the same action on a different space:

$$\Omega_q T^*N = \{\gamma\colon I \to T^*N \mid \pi(\gamma(0)) = \pi(\gamma(1)) = q\}.$$

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- It is also well known that critical points of the action defined on $\Omega_q T^*N$ are: time 1 flow curves for the Hamiltonian flow starting and ending in the fiber.
- We denote this Floer homology by $FH_*^a(H, q)$ for a regular value a.

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- Since we are dealing with paths and not loops we get an action "distortion":

$$A_H(\gamma) = \int_\gamma \lambda - Hdt = \int_\gamma \lambda' - Hdt + f(\gamma(0)) - f(\gamma(1)),$$

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- So for a critical point γ (for A_{H_s} defined on $\Omega_q T^*N$) with critical value less than $-K$ we know precisely as before that increasing s similarly decreases the critical value, and we may thus define

$$SH_*(M, q) = \operatorname*{colim}_{s\to\infty} FH_*^a(H_s, q)$$

  for any $a < -K$.

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Lemma

The homologies $SH_*(M, q)$ *are independent of* q *and form a local system* $SH_*(M, \bullet)$ *of graded abelian groups on* N*.*

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Theorem (K)

There is a Serre type spectral sequence strongly converging to $SH_*(M)$ with second page isomorphic to $H_*(N, SH_*(M, \bullet))$.

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- To see how, we look at the critical points of A_{H_s} restricted to $\Omega_{\beta(v)} T^*N$ for $v \in I$. This defines a socalled bifurcation diagram.

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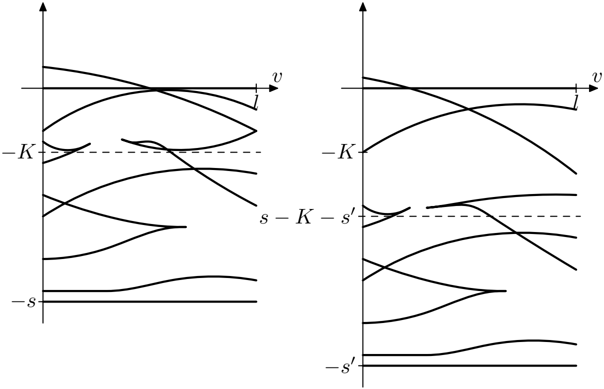

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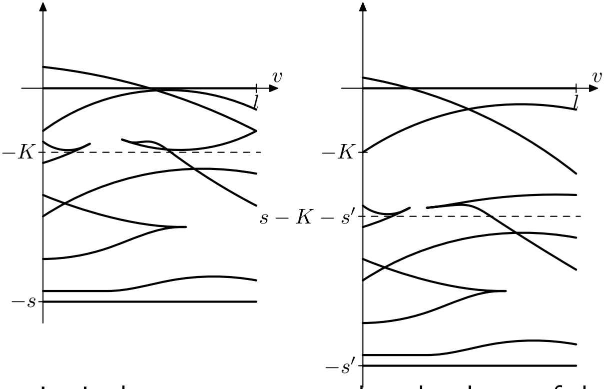

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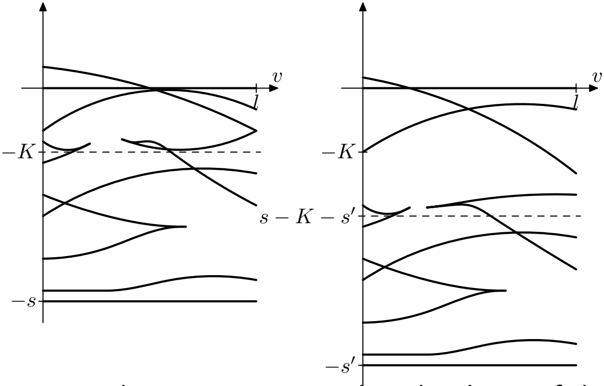

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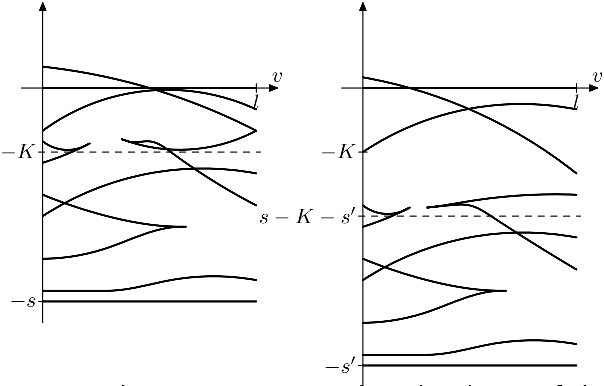

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- Taking the colimit as $s \to \infty$ then defines the map

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- Finally one may prove that this map only depends on the homotopy type of β fixing the end points.

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- So I construct a fibration of spectra, and these naturally have a serre spectral sequence.