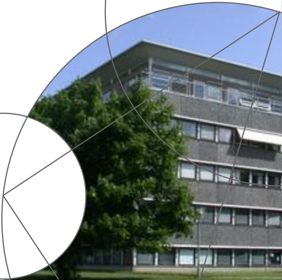
Faculty of Science

A Distribution Result Related to Automorphic Forms

Flemming von Essen
Department of Mathematical Sciences



Fuchsian groups

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ let

$$\gamma z = \frac{az + b}{cz + d}.$$

Fuchsian groups

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ let

$$\gamma z = \frac{az+b}{cz+d}.$$

We will consider Fuchsian groups, i.e. discrete subgroups of $SL_2(\mathbb{R})$.

Fuchsian groups

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ let

$$\gamma z = \frac{az+b}{cz+d}.$$

We will consider Fuchsian groups, i.e. discrete subgroups of $SL_2(\mathbb{R})$.
Let Γ be such a group. We say that $\gamma \in \Gamma \backslash \{\pm I\}$ is

- elliptic if $|Tr\gamma| < 2$ (or if γ fixes a point in the upper half plane $\mathbb{H}$).
- parabolic if $|Tr\gamma| = 2$ (or if γ fixes one point in $\mathbb{R} \cup \{\infty\}$).
- hyperbolic if $|Tr\gamma| > 2$ (or if γ fixes two points in $\mathbb{R} \cup \{\infty\}$).

Hyperbolic matrices and geodesics

Let γ be a hyperbolic (i.e. $|Tr\gamma| > 2$). Then there exists $|\lambda| > 1$ and $A \in SL_2(\mathbb{R})$ s.t.

$$\gamma = A \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} A^{-1},$$

and we define $N(\gamma) := \lambda^2$ and $l(\gamma) = \log N(\gamma)$.

Hyperbolic matrices and geodesics

Let γ be a hyperbolic (i.e. $|Tr\gamma| > 2$). Then there exists $|\lambda| > 1$ and $A \in SL_2(\mathbb{R})$ s.t.

$$\gamma = A \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} A^{-1},$$

and we define $N(\gamma) := \lambda^2$ and $l(\gamma) = \log N(\gamma)$.
Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$. Then $\Gamma\backslash\mathbb{H}$ is a Riemann surface, and there is a bijection between conjugacy classes $\{\gamma\} = \{\tau\gamma\tau^{-1} \mid \tau \in \Gamma\}$ of hyperbolic elements in Γ and closed geodesics on $\Gamma\backslash\mathbb{H}$.

Hyperbolic matrices and geodesics

Let γ be a hyperbolic (i.e. $|Tr\gamma| > 2$). Then there exists $|\lambda| > 1$ and $A \in SL_2(\mathbb{R})$ s.t.

$$\gamma = A \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} A^{-1},$$

and we define $N(\gamma) := \lambda^2$ and $l(\gamma) = \log N(\gamma)$.
Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$. Then $\Gamma\backslash\mathbb{H}$ is a Riemann surface, and there is a bijection between conjugacy classes $\{\gamma\} = \{\tau\gamma\tau^{-1} \mid \tau \in \Gamma\}$ of hyperbolic elements in Γ and closed geodesics on $\Gamma\backslash\mathbb{H}$.
The geodesic associated with γ has length $l(\gamma)$.

Automorphic forms

Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$, and $f : \mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\} \to \mathbb{C}$ be holomorphic with

$$f(\gamma z) = f\left(\frac{az+b}{cz+d}\right) = \qquad (cz+d)^k f(z),$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and some $k \in \mathbb{R}$. We say that f is an (classical) automorphic form wrt. Γ of weight k.

Automorphic forms

Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$, and $f : \mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\} \to \mathbb{C}$ be holomorphic with

$$f(\gamma z) = f\left(\frac{az+b}{cz+d}\right) = \nu(\gamma)(cz+d)^k f(z),$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and some $k \in \mathbb{R}$. We say that f is an (classical) automorphic form wrt. Γ of weight k with multiplier system ν.

Automorphic forms

Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$, and $f : \mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\} \to \mathbb{C}$ be holomorphic with

$$f(\gamma z) = f\left(\frac{az+b}{cz+d}\right) = \nu(\gamma)(cz+d)^k f(z),$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and some $k \in \mathbb{R}$. We say that f is an (classical) automorphic form wrt. Γ of weight k with multiplier system ν. If ν is a multiplier system on Γ the following should hold

1) $|\nu(\gamma)| = 1$, for all $\gamma \in \Gamma$,
2) $\nu(-I) = exp(-\pi i k)$ if $-I \in \Gamma$,
3) $\nu(\gamma_1\gamma_2) = \sigma_k(\gamma_1, \gamma_2)\nu(\gamma_1)\nu(\gamma_2)$.

Zero free automorphic forms

Let $f : \mathbb{H} \to \mathbb{C}$ be a zero free automorphic form, so

$$f\left(\frac{az+b}{cz+d}\right) = \nu(\gamma)(cz+d)^k f(z),$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Zero free automorphic forms

Let $f : \mathbb{H} \to \mathbb{C}$ be a zero free automorphic form, so

$$f\left(\frac{az+b}{cz+d}\right) = \nu(\gamma)(cz+d)^k f(z),$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. We can take a holomorphic logarithm

$$\log f\left(\frac{az+b}{cz+d}\right) = 2\pi i k \Phi(\gamma) + k \log(cz+d) + \log f(z), \quad (1)$$

where $\exp(2\pi i k \Phi(\gamma)) = \nu(\gamma)$.

Zero free automorphic forms

Let $f : \mathbb{H} \to \mathbb{C}$ be a zero free automorphic form, so

$$f\left(\frac{az+b}{cz+d}\right) = \nu(\gamma)(cz+d)^k f(z),$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. We can take a holomorphic logarithm

$$\log f\left(\frac{az+b}{cz+d}\right) = 2\pi i k\Phi(\gamma) + k\log(cz+d) + \log f(z), \quad (1)$$

where $\exp(2\pi i k\Phi(\gamma)) = \nu(\gamma)$.
Multiplying with $m \in \mathbb{R}$ in (1) and taking the exponential gives us a m'th power of f, which is an automorphic form of weight km, and multiplier system $\exp(2\pi i km\Phi(\gamma))$.

The Dedekind η-function

Let $\eta : \mathbb{H} \to \mathbb{C}$ be defined by

$$\eta(z) = \exp\left(\frac{\pi i z}{12}\right) \prod_{n=1}^{\infty} (1 - \exp(2\pi i n z)),$$

then η is a zero free weight $1/2$ automorphic form on $SL_2(\mathbb{Z})$.

The Dedekind η-function

Let $\eta : \mathbb{H} \to \mathbb{C}$ be defined by

$$\eta(z) = \exp\left(\frac{\pi i z}{12}\right) \prod_{n=1}^{\infty} (1 - \exp(2\pi i n z)),$$

then η is a zero free weight $1/2$ automorphic form on $SL_2(\mathbb{Z})$.Taking logarithms as on the previous slide we get

$$(\log \eta)\left(\frac{az+b}{cz+d}\right) = \pi i \Phi(\gamma) + \log(cz+d)/2 + (\log \eta)(z),$$

where 12Φ is the so called Rademacher function.

Ghys' theorem

$\Gamma\backslash\mathbb{H}$ is homeomorphic to $\{(x,y) \in C^2 \mid |x|^2 + |y|^2 = 1\}\backslash\tau$, where τ is a trefoil knot.

Ghys' theorem

$\Gamma\backslash\mathbb{H}$ is homeomorphic to $\{(x,y) \in C^2 \mid |x|^2 + |y|^2 = 1\}\backslash\tau$, where τ is a trefoil knot.

Theorem (É Ghys)

For $\gamma \in SL_2(\mathbb{Z})$ hyperbolic, there is a (oriented) curve γ' in $S^3\backslash\tau$ associated with γ, and $12\Phi(\gamma)$ is the linking number between γ' and τ.

$$(\log\eta)\left(\frac{az+b}{cz+d}\right) = \pi i\Phi(\gamma) + \log(cz+d)/2 + (\log\eta)(z),$$

Ghys' theorem

$\Gamma\backslash\mathbb{H}$ is homeomorphic to $\{(x, y) \in C^2 \mid |x|^2 + |y|^2 = 1\}\backslash\tau$, where τ is a trefoil knot.

Theorem (É Ghys)

For $\gamma \in SL_2(\mathbb{Z})$ hyperbolic, there is a (oriented) curve γ' in $S^3\backslash\tau$ associated with γ, and $12\Phi(\gamma)$ is the linking number between γ' and τ.

$$(\log\eta)\left(\frac{az+b}{cz+d}\right) = \pi i\Phi(\gamma) + \log(cz+d)/2 + (\log\eta)(z),$$

The number of times the curves wind around each other.

Distribution of the geodesics

P. Sarnak and C. J. Mozzochi has showed that if Φ is as on the previous slides. So Φ is related to $SL_2(\mathbb{Z})$ and η.

Then

$$\lim_{T\to\infty} \frac{\#\{\{\gamma\} \mid l(\gamma) \leq T, a \leq \Phi(\gamma)/l(\gamma) \leq b\}}{\#\{\{\gamma\} \mid l(\gamma) \leq T\}} = \frac{\arctan(4\pi b) - \arctan(4\pi a)}{\pi}$$

where for $|Tr\gamma| > 2$, $\gamma \neq \tau^n$ and

$$\{\gamma\} = \{\tau\gamma\tau^{-1} \mid \tau \in SL_2(\mathbb{Z})\}.$$

Distribution of the geodesics

P. Sarnak and C. J. Mozzochi has showed that if Φ is as on the previous slides. So Φ is related to $SL_2(\mathbb{Z})$ and η.

Then

$$\lim_{T\to\infty} \frac{\#\{\{\gamma\} \mid l(\gamma) \leq T, a \leq \Phi(\gamma)/l(\gamma) \leq b\}}{\#\{\{\gamma\} \mid l(\gamma) \leq T\}} = \frac{\arctan(4\pi b) - \arctan(4\pi a)}{\pi}$$

where for $|Tr\gamma| > 2$, $\gamma \neq \tau^n$ and

$$\{\gamma\} = \{\tau\gamma\tau^{-1} \mid \tau \in SL_2(\mathbb{Z})\}.$$

Can this be generalized to other groups?

Distribution of the geodesics

P. Sarnak and C. J. Mozzochi has showed that if Φ is as on the previous slides. So Φ is related to $SL_2(\mathbb{Z})$ and η.

Then

$$\lim_{T\to\infty} \frac{\#\{\{\gamma\} \mid l(\gamma) \leq T, a \leq \Phi(\gamma)/l(\gamma) \leq b\}}{\#\{\{\gamma\} \mid l(\gamma) \leq T\}} = \frac{\arctan(4\pi b) - \arctan(4\pi a)}{\pi}$$

where for $|Tr\gamma| > 2$, $\gamma \neq \tau^n$ and

$$\{\gamma\} = \{\tau\gamma\tau^{-1} \mid \tau \in SL_2(\mathbb{Z})\}.$$

Can this be generalized to other groups? Yes.

Distribution of the geodesics

Let f be a zero free automorphic form on a Fuchsian group Γ, such that $\mu(\Gamma\backslash\mathbb{H}) < \infty$, $\Phi : \Gamma \to \mathbb{Q}$ and

$$\log f\left(\frac{az+b}{cz+d}\right) = 2\pi ik\Phi(\gamma) + k\log(cz+d) + \log f(z).$$

Then

$$\lim_{T\to\infty} \frac{\#\{\{\gamma\} \mid l(\gamma) \leq T, a \leq \Phi(\gamma)/l(\gamma) \leq b\}}{\#\{\{\gamma\} \mid l(\gamma) \leq T\}} = \frac{\arctan(4\pi b) - \arctan(4\pi a)}{\pi}$$

where for $|Tr\gamma| > 2$, $\gamma \neq \tau^n$ and

$$\{\gamma\} = \{\tau\gamma\tau^{-1} \mid \tau \in \Gamma\}.$$

Examples of zero free automorphic forms

Let $n = 1, 2, 3, 4$, then $\eta\left(\frac{z}{\sqrt{n}}\right)\eta(\sqrt{n}z)$ is a zero free automorphic form, wrt. the group generated by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & \sqrt{n} \\ 0 & 1 \end{pmatrix}.$$

Examples of zero free automorphic forms

Let $n = 1, 2, 3, 4$, then $\eta\left(\frac{z}{\sqrt{n}}\right)\eta(\sqrt{n}z)$ is a zero free automorphic form, wrt. the group generated by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & \sqrt{n} \\ 0 & 1 \end{pmatrix}.$$

$\theta(z) = \sum_{n\in\mathbb{Z}} \exp(2\pi i n^2 z) = \frac{\eta(2z)^5}{\eta(z)^2\eta(4z)^2}$
$\theta_M(z) = \sum_{n\in\mathbb{Z}} (-1)^n \exp(2\pi i n^2 z) = \frac{\eta(z)^2}{\eta(2z)}$
and $\theta_F(z) = \sum_{n\in\mathbb{Z}} \exp(2\pi i (n+1/2)^2 z) = 2\frac{\eta(4z)^2}{\eta(2z)}$ are automorphic forms wrt.

$$\Gamma_0(4) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \gamma \in SL_2(\mathbb{Z}),\, 4 \mid c \right\}.$$

Distribution of the geodesics

We want to prove that.

Let f be a zero free automorphic form on a Fuchsian group Γ, such that $\mu(\Gamma\backslash\mathbb{H}) < \infty$, $\Phi : \Gamma \to \mathbb{Q}$ and

$$\log f\left(\frac{az+b}{cz+d}\right) = 2\pi ik\Phi(\gamma) + k\log(cz+d) + \log f(z).$$

Then

$$\lim_{T\to\infty} \frac{\#\{\{\gamma\} \mid l(\gamma) \leq T, a \leq \Phi(\gamma)/l(\gamma) \leq b\}}{\#\{\{\gamma\} \mid l(\gamma) \leq T\}} = \frac{\arctan(4\pi b) - \arctan(4\pi a)}{\pi}$$

where for $|Tr\gamma| > 2$, $\gamma \neq \tau^n$ and

$$\{\gamma\} = \{\tau\gamma\tau^{-1} \mid \tau \in \Gamma\}.$$

Idea of the proof - The operator Δ_k

If f is a classical automorphic form of weight k, we can create another type of automorphic form f^* given by $f^*(z) = f(z)(\Im z)^{k/2}$, which transforms in the following way

$$f^*\left(\frac{az+b}{cz+d}\right) = \nu(\gamma)\left(\frac{cz+d}{|cz+d|}\right)^k f^*(z).$$

Idea of the proof - The operator Δ_k

If f is a classical automorphic form of weight k, we can create another type of automorphic form f^* given by $f^*(z) = f(z)(\Im z)^{k/2}$, which transforms in the following way

$$f^*\left(\frac{az+b}{cz+d}\right) = \nu(\gamma)\left(\frac{cz+d}{|cz+d|}\right)^k f^*(z).$$

Then is f^* a eigenfunction of Δ_k given by

$$\Delta_k = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - iky\frac{\partial}{\partial x}.$$

Selberg's trace formula

Theorem (Selberg's trace formula)

If g a smooth function that decreases sufficiently quick, and h is the inverse Fourier transform of g, then

$$\sum_{n=0}^{\infty} h(r_n) = \sum_{\substack{\{\gamma\} \\ Tr\gamma>2}} \frac{\nu(\gamma) l(\gamma_0)}{N(\gamma)^{1/2} - N(\gamma)^{-1/2}} g(l(\gamma))$$

$$+\text{some other terms.}$$

Selberg's trace formula

Theorem (Selberg's trace formula)

If g a smooth function that decreases sufficiently quick, and h is the inverse Fourier transform of g, then

$$\sum_{n=0}^{\infty} h(r_n) = \sum_{\substack{\{\gamma\} \\ Tr\gamma > 2}} \frac{\nu(\gamma) l(\gamma_0)}{N(\gamma)^{1/2} - N(\gamma)^{-1/2}} g(l(\gamma))$$

$$+\text{some other terms.}$$

Here the $r_n^2 + 1/4$ is the eigenvalues of Δ_k.

Selberg's trace formula

Theorem (Selberg's trace formula)

If g a smooth function that decreases sufficiently quick, and h is the inverse Fourier transform of g, then

$$\sum_{n=0}^{\infty} h(r_n) = \sum_{\substack{\{\gamma\} \\ Tr\gamma>2}} \frac{\nu(\gamma) l(\gamma_0)}{N(\gamma)^{1/2} - N(\gamma)^{-1/2}} g(l(\gamma)) + \text{some other terms.}$$

Here the $r_n^2 + 1/4$ is the eigenvalues of Δ_k.
To prove the theorem we use a family of g's that is approximately indicator functions, and multiplier systems on the form $\nu(\gamma) = \exp(2\pi i k \Phi(\gamma))$, for arbitrary k. Then we do a lot of estimations on the terms in the formula.

The eigenvalues

We want to estimate the contribution from the eigenvalues

$$\sum_{n=0}^{\infty} h(r_n).$$

The eigenvalues

We want to estimate the contribution from the eigenvalues

$$\sum_{n=0}^{\infty} h(r_n).$$

- It turns out that we only need to consider small eigenvalues, since h is decreasing rapidly.

The eigenvalues

We want to estimate the contribution from the eigenvalues

$$\sum_{n=0}^{\infty} h(r_n).$$

- It turns out that we only need to consider small eigenvalues, since h is decreasing rapidly.
- For small weight m, the smallest eigenvalue comes from our original $(f^*)^{m/k}$, where f is our zero free classical automorphic form. This eigenvalue is $\frac{|m|}{2}\left(1-\frac{|m|}{2}\right)$.

The eigenvalues

We want to estimate the contribution from the eigenvalues

$$\sum_{n=0}^{\infty} h(r_n).$$

- It turns out that we only need to consider small eigenvalues, since h is decreasing rapidly.
- For small weight m, the smallest eigenvalue comes from our original $(f^*)^{m/k}$, where f is our zero free classical automorphic form. This eigenvalue is $\frac{|m|}{2}\left(1 - \frac{|m|}{2}\right)$.
- We need to show that for small weight, this eigenvalue has multiplicity 1 and the next eigenvalue is bounded from below.

The eigenvalues

We want to estimate the contribution from the eigenvalues

$$\sum_{n=0}^{\infty} h(r_n).$$

- It turns out that we only need to consider small eigenvalues, since h is decreasing rapidly.
- For small weight m, the smallest eigenvalue comes from our original $(f^*)^{m/k}$, where f is our zero free classical automorphic form. This eigenvalue is $\frac{|m|}{2}\left(1-\frac{|m|}{2}\right)$.
- We need to show that for small weight, this eigenvalue has multiplicity 1 and the next eigenvalue is bounded from below.
- For weight 0, the multiplicity is 1. So it is enough to show that, the eigenvalues are continuous in 0.

End of the sketched proof

The lower bound on the eigenvalues, enables us to estimate the $\sum_n h(r_n)$. This along with estimates on the rest of the terms in the trace formula and summation by parts gives us

$$\sum_{\substack{\{\gamma\} \\ N(\gamma)\leq x}} \nu(\gamma)I(\gamma) = \frac{x^{1-|k|/2}}{1-|k|/2}\mathbf{1}_{[0,k_\delta]}(|k|) + O\left(x^{1-\delta_\Gamma}\log\frac{1}{|k|}\right).$$

End of the sketched proof

The lower bound on the eigenvalues, enables us to estimate the $\sum_n h(r_n)$. This along with estimates on the rest of the terms in the trace formula and summation by parts gives us

$$\sum_{\substack{\{\gamma\} \\ N(\gamma)\leq x}} \nu(\gamma) l(\gamma) = \frac{x^{1-|k|/2}}{1-|k|/2} \mathbf{1}_{[0,k_\delta]}(|k|) + O\left(x^{1-\delta_\Gamma} \log \frac{1}{|k|}\right).$$

Here $\nu(\gamma) = \exp(2\pi i k \Phi(\gamma))$, so if we multiply with $\exp(2\pi i k n)$ and integrate, we can estimate

$$\sum_{\substack{\{\gamma\} \\ N(\gamma)\leq x \\ \Phi(\gamma)=n}} l(\gamma) = c_\Gamma \int_2^x \frac{\ln y}{(4\pi n/N)^2 + (\ln y)^2}\, dy + O(x^{1-\delta_\Gamma}).$$

End of the sketched proof

The lower bound on the eigenvalues, enables us to estimate the $\sum_n h(r_n)$. This along with estimates on the rest of the terms in the trace formula and summation by parts gives us

$$\sum_{\substack{\{\gamma\} \\ N(\gamma)\leq x}} \nu(\gamma) l(\gamma) = \frac{x^{1-|k|/2}}{1-|k|/2} \mathbf{1}_{[0,k_\delta]}(|k|) + O\left(x^{1-\delta_\Gamma} \log \frac{1}{|k|}\right).$$

Here $\nu(\gamma) = \exp(2\pi i k \Phi(\gamma))$, so if we multiply with $\exp(2\pi i k n)$ and integrate, we can estimate

$$\sum_{\substack{\{\gamma\} \\ N(\gamma)\leq x \\ \Phi(\gamma)=n}} l(\gamma) = c_\Gamma \int_2^x \frac{\ln y}{(4\pi n/N)^2 + (\ln y)^2} \, dy + O(x^{1-\delta_\Gamma}).$$

Working with this we get our distribution result.

Interpretations

We proved

$$\lim_{T\to\infty} \frac{\#\{\{\gamma\} \mid l(\gamma) \leq T, a \leq \Phi(\gamma)/l(\gamma) \leq b\}}{\#\{\{\gamma\} \mid l(\gamma) \leq T\}} = \frac{\arctan(4\pi b) - \arctan(4\pi a)}{\pi}.$$

Interpretations

We proved

$$\lim_{T\to\infty} \frac{\#\{\{\gamma\} \mid l(\gamma) \leq T, a \leq \Phi(\gamma)/l(\gamma) \leq b\}}{\#\{\{\gamma\} \mid l(\gamma) \leq T\}} = \frac{\arctan(4\pi b) - \arctan(4\pi a)}{\pi}.$$

For $\Gamma = SL_2(\mathbb{Z})$, this was interesting due to Ghys' theorem

Theorem (É Ghys)

For $\gamma \in SL_2(\mathbb{Z})$ hyperbolic, there is a (oriented) curve γ' in $S^3\backslash\tau$ associated with γ, and $12\Phi(\gamma)$ is the linking number between γ' and τ.

Interpretations

We proved

$$\lim_{T\to\infty} \frac{\#\{\{\gamma\} \mid l(\gamma) \leq T, a \leq \Phi(\gamma)/l(\gamma) \leq b\}}{\#\{\{\gamma\} \mid l(\gamma) \leq T\}} = \frac{\arctan(4\pi b) - \arctan(4\pi a)}{\pi}.$$

For $\Gamma = SL_2(\mathbb{Z})$, this was interesting due to Ghys' theorem

Theorem (É Ghys)

For $\gamma \in SL_2(\mathbb{Z})$ hyperbolic, there is a (oriented) curve γ' in $S^3\backslash\tau$ associated with γ, and $12\Phi(\gamma)$ is the linking number between γ' and τ.

Is there an interpretation of Φ for other groups?

Interpretations

We proved

$$\lim_{T\to\infty} \frac{\#\{\{\gamma\} \mid l(\gamma) \leq T, a \leq \Phi(\gamma)/l(\gamma) \leq b\}}{\#\{\{\gamma\} \mid l(\gamma) \leq T\}} = \frac{\arctan(4\pi b) - \arctan(4\pi a)}{\pi}.$$

For $\Gamma = SL_2(\mathbb{Z})$, this was interesting due to Ghys' theorem

Theorem (É Ghys)

For $\gamma \in SL_2(\mathbb{Z})$ hyperbolic, there is a (oriented) curve γ' in $S^3\backslash\tau$ associated with γ, and $12\Phi(\gamma)$ is the linking number between γ' and τ.

Is there an interpretation of Φ for other groups?
Yes (at least for some groups).