A flow approach to special holonomy

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Talk based on

- *A heat flow for special metrics*
  joint with F. Witt, Adv. Math. 231, 2012, no. 6
- *Energy functionals and soliton equations for G_2-forms*
  joint with F. Witt, Ann. Global Anal. Geom. 42, 2012, no. 4
- *A spinorial energy functional: critical points and gradient flow*
  joint with B. Ammann and F. Witt, arXiv:1207.3529
- *The spinor flow on surfaces*
  joint with B. Ammann and F. Witt, in preparation

The holonomy group of a Riemannian manifold

Definition

Let (M, g) be a Riemannian manifold. The Riemannian metric g determines the Levi-Civita connection

$$\nabla^g : \Gamma(M, TM) \to \Gamma(M, T^*M \otimes TM), Y \mapsto (X \mapsto \nabla^g_X Y)$$

and hence a linear isometry

$$P_\gamma : T_{x_0}M \to T_{x_1}M$$

for any path γ from x_0 to x_1, the *parallel transport* along γ. Then

$$\mathrm{Hol}(M, g) := \{P_\gamma \in \mathrm{O}(T_{x_0}M) : \gamma \text{ loop in } x_0\}$$

is the *holonomy group* of (M, g) and

$$\begin{aligned}\mathrm{Hol}_0(M, g) :=& \{P_\gamma \in \mathrm{SO}(T_{x_0}M) : \gamma \text{ nullhomotopic}\} \\ &\subset \mathrm{Hol}(M, g)\end{aligned}$$

the *reduced* holonomy group.

The holonomy group of a Riemannian manifold

Berger's list

If (M, g) irreducible, simply-connected and non-symmetric, then $\mathrm{Hol}(M, g)$ is one of the following:

$\mathrm{Hol}(M, g)$	$\dim M$	geometry
$\mathrm{SO}(n)$	n	generic
$\mathrm{U}(m)$	$2m$	Kähler
$\mathrm{SU}(m)$	$2m$	Calabi-Yau (Ricci-flat)
$\mathrm{Sp}(k)$	$4k$	hyperkähler (Ricci-flat)
$\mathrm{Sp}(1)\mathrm{Sp}(k)$	$4k$	quaternion-Kähler (Einstein)
G_2	7	G_2 (Ricci-flat)
Spin_7	8	Spin_7 (Ricci-flat)

Hitchin: $\exists$ parallel unit spinor $\Leftrightarrow$ g Ricci-flat and of special holonomy $\Leftrightarrow$ $\mathrm{Hol}(M, g) = \mathrm{SU}(m), \mathrm{Sp}(k), \mathrm{G}_2$ or Spin_7

The spinorial energy functional

Definition

Let M be a compact spin manifold, $n = \dim M \geq 2$.

- g a Riemannian metric $\rightsquigarrow \Sigma_g M \to M$, the complex g-spinor bundle, typical fiber: the complex spinor module Σ_n
- $\Sigma M \to M$ the *universal spinor bundle*, typical fiber: the vector bundle $(\widetilde{\mathrm{GL}}_n^+ \times \Sigma_n)/\mathrm{Spin}_n \to \widetilde{\mathrm{GL}}_n^+/\mathrm{Spin}_n \cong \odot_+^2 \mathbb{R}^{n*}$, which carries a connection, the *Bourguignon-Gauduchon connection*
- $1:1$ Correspondence

$$\Phi \in \Gamma(\Sigma M) \longleftrightarrow g \in \Gamma(\odot_+^2 T^*M), \varphi \in \Gamma(\Sigma_g M)$$

- g_t path of Riemannian metrics $\rightsquigarrow$ horizontal lift $\Phi_t = (g_t, \varphi_t) \in \Gamma(\Sigma M)$ using Bourguignon-Gauduchon
- Geometric interpretation of parallel transport provided by generalized cylinder construction of Bär-Gauduchon-Moroianu

The spinorial energy functional
Definition

Let M be a compact spin manifold, $n = \dim M \geq 2$.

- $\langle \cdot\,,\cdot \rangle = \mathrm{Re}\, h(\cdot\,,\cdot)$ real inner product on spinors
- $S(\Sigma M) = \{\Phi_x = (g_x, \varphi_x) \in \Sigma M : \langle \varphi_x, \varphi_x \rangle = 1\}$
- $\mathcal{N} = \Gamma(S(\Sigma M))$, the space of unit spinors

We consider the energy functional

$$\begin{aligned} \mathcal{E} : \mathcal{N} &\longrightarrow \mathbb{R}_{\geq 0} \\ \Phi &\longmapsto \frac{1}{2} \int_M |\nabla^g \varphi|_g^2 \, dv_g \end{aligned}$$

where $\Phi = (g, \varphi)$ as above

The spinorial energy functional

Symmetries

- Diffeomorphism Invariance

$$F \text{ spin-diffeomorphism} \Rightarrow \mathcal{E}(F_* \Phi) = \mathcal{E}(\Phi)$$

- Scaling

$$\lambda \in \mathbb{R} \Rightarrow \mathcal{E}(\lambda^2 g, \varphi) = \lambda^{n-2} \mathcal{E}(g, \varphi)$$

- Representation theory

$$L : \Sigma_n \to \Sigma_n \ \mathrm{Spin}_n\text{-equivariant isometry} \Rightarrow \mathcal{E}(g, L(\varphi)) = \mathcal{E}(g, \varphi)$$

Example: $\Sigma_n = \Sigma_n^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \Leftrightarrow \exists$ real structure $J : \Sigma_n \to \Sigma_n$

The spinorial energy functional

The gradient

For $(g, \varphi) \in \mathcal{N}$ consider the subbundle

$$\varphi^\perp = \{\dot{\varphi}_x \in \Sigma_g M : \langle \varphi_x, \dot{\varphi}_x \rangle = 0\}$$

Using the Gauduchon-Bourguignon connection we split

$$T_{(g,\varphi)}\mathcal{N} = \Gamma(\odot^2 T^* M) \oplus \Gamma(\varphi^\perp)$$

Consider negative gradient of $\mathcal{E} : \mathcal{N} \to \mathbb{R}$ in L^2-sense

$$-\operatorname{grad} \mathcal{E}(g, \varphi) =: Q(g, \varphi) = (Q_1(g, \varphi), Q_2(g, \varphi))$$

with $Q_1(g, \varphi) \in \Gamma(\odot^2 T^* M)$ and $Q_2(g, \varphi) \in \Gamma(\varphi^\perp)$, i.e.

$$-D_{g,\varphi}\mathcal{E}(\dot{g}, \dot{\varphi}) = \int_M (Q_1(g, \varphi)_g, \dot{g})_g + \langle Q_2(g, \varphi), \dot{\varphi} \rangle \, dv_g$$

The spinorial energy functional

The gradient

Theorem (Ammann-W.-Witt)

$$Q_1(g,\varphi) = -\tfrac{1}{4}|\nabla^g\varphi|_g^2\, g - \tfrac{1}{4}\mathrm{div}_g\, T_{g,\varphi} + \tfrac{1}{2}\langle\nabla^g\varphi\otimes\nabla^g\varphi\rangle$$
$$Q_2(g,\varphi) = -\nabla^{g*}\nabla^g\varphi + |\nabla^g\varphi|_g^2\,\varphi$$

where

- $T_{g,\varphi}\in\Gamma(T^*M\otimes\odot^2 T^*M)$ *is the symmetrization in the second and third component of the 3-tensor*

$$(X,Y,Z)\mapsto\langle(X\wedge Y)\cdot\varphi,\nabla^g_Z\varphi\rangle$$

- $\langle\nabla^g\varphi\otimes\nabla^g\varphi\rangle$ *is the symmetric 2-tensor defined by*

$$(X,Y)\mapsto\langle\nabla^g_X\varphi,\nabla^g_Y\varphi\rangle$$

The spinorial energy functional

Critical points ($n \geq 3$)

Taking the trace of the first component yields

$$-4\,\mathrm{Tr}_g\, Q_1(g,\varphi) = \mathrm{Tr}_g \operatorname{div}_g T_{g,\varphi} + (n-2)|\nabla^g\varphi|_g^2,$$

in particular

$$-4\int_M \mathrm{Tr}_g\, Q_1(g,\varphi)\, dv_g = (n-2)\int_M |\nabla^g\varphi|_g^2\, dv_g.$$

Corollary

Let $n \geq 3$. Then (g,φ) is critical $\Leftrightarrow \nabla^g\varphi = 0$, in particular g is Ricci-flat and of special holonomy.

φ is a g-Killing spinor with constant $\lambda \in \mathbb{R}$ if $\nabla^g_X\varphi = \lambda X \cdot \varphi$ for all $X \in \Gamma(TM)$. Killing spinors are critical points under the constraint $\mathrm{vol}(M,g) = 1$.

The spinorial energy functional

Critical points ($n = 2$)

The functional is scale invariant in this dimension. Hence

$$(g, \varphi) \text{ critical point} \Leftrightarrow (g, \varphi) \text{ constrained critical point}$$

Theorem (Ammann-W.-Witt)

Let $n = 2$. Then

- $\chi(M) > 0$*: (g, φ) is critical $\Leftrightarrow$ (g, φ) is a global minimum $\Leftrightarrow$ $\varphi = \cos\vartheta\,\psi + \sin\vartheta\,\omega \cdot \psi$ for a g-Killing spinor ψ (ω the real volume element, $\vartheta \in \mathbb{R}$)*
- $\chi(M) = 0$*: (g, φ) is a global minimum $\Leftrightarrow$ $\nabla^g \varphi = 0$*
- $\chi(M) < 0$*: (g, φ) is a global minimum $\Leftrightarrow$ $D_g \varphi = 0$*

The spinor flow

Short-time existence and uniqueness

Consider the *spinor flow* with initial condition $\Phi \in \mathcal{N}$

$$\partial_t \Phi_t = Q(\Phi_t), \quad \Phi_0 = \Phi$$

for time-dependent family $\Phi_t = (g_t, \varphi_t) \in \mathcal{N}$, $t \geq 0$.

Theorem (Ammann-W.-Witt)

The spinor flow has a unique short-time solution.

Uniqueness implies: All symmetries are preserved under the flow.

Ingredients of proof:

- $\sigma_\xi(D_\Omega Q) \geq 0$ for all $\xi \in T^*M$
- $\ker \sigma_\xi(D_\Omega Q)$ precisely coming from diffeomorphism invariance
- DeTurck trick: $\tilde{Q}(\Phi) := Q(\Phi) + \mathcal{L}_{X(\Phi)}\Phi$ for $X(\Phi)$ a cleverly chosen vector field

G_2-geometry

Locally

Let

$$\Omega_0 := e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245} \in \Lambda^3\mathbb{R}^{7*}$$

where $e^{ijk} := e^i \wedge e^j \wedge e^k$. Then

$$\mathrm{G}_2 := \{A \in \mathrm{GL}_7 : A^*\Omega_0 = \Omega_0\} \subset \mathrm{SO}(7)$$

i.e. G_2 preserves Euclidean metric and standard orientation on $\mathbb{R}^7$.

$$\Lambda^3_+\mathbb{R}^{7*} := \mathrm{GL}_7^+\text{-orbit of } \Omega_0$$
$$\cong \mathrm{GL}_7^+/\mathrm{G}_2$$

The orbit $\Lambda^3_+\mathbb{R}^{7*} \subset \Lambda^3\mathbb{R}^{7*}$ is

- open ($\dim \Lambda^3\mathbb{R}^{7*} = 35 = 49 - 14 = \dim \mathrm{GL}_7^+ - \dim \mathrm{G}_2$)
- a positive cone ($\Omega \in \Lambda^3_+\mathbb{R}^{7*}, \lambda > 0 \Rightarrow \lambda\Omega \in \Lambda^3_+\mathbb{R}^{7*}$)

G_2-geometry
Globally

Let M^7 be compact and oriented. Set

$$\Lambda^3_+ T^*M := P_{\mathrm{GL}_7^+} \times_{\mathrm{GL}_7^+} \Lambda^3_+ \mathbb{R}^{7*}$$

A section $\Omega \in \Gamma(M, \Lambda^3_+ T^*M) =: \Omega^3_+(M)$ is called *positive* 3-form.

$$\Omega \in \Omega^3_+(M) \longleftrightarrow \text{reduction of structure group of } TM \text{ from } \mathrm{GL}_7^+ \text{ to } \mathrm{G}_2 \subset \mathrm{SO}(7)$$

In particular: $\Omega \rightsquigarrow$ metric quantities $g_\Omega, \star_\Omega, \mathrm{vol}_\Omega, \ldots$

$$\mathrm{Hol}(M, g_\Omega) \subset \mathrm{G}_2 \underset{\textit{Fernandez, Gray}}{\Longleftrightarrow} d\Omega = d\star_\Omega \Omega = 0$$

$$\underset{\textit{Bonan}}{\Longrightarrow} \mathrm{Ric}^{g_\Omega} = 0$$

$\Omega \in \Omega^3_+(M)$ satisfying $d\Omega = d\star_\Omega \Omega = 0$ is called *torsion-free*.

The G_2-flow

Definition

Let M be a compact, oriented 7-manifold and $\Omega^3_+(M)$ the space of positive 3-forms on M. Consider

$$\begin{aligned} \mathcal{D} : \Omega^3_+(M) &\longrightarrow \mathbb{R}_{\geq 0} \\ \Omega &\longmapsto \tfrac{1}{2} \int_M \{ |d\Omega|^2_\Omega + |d \star_\Omega \Omega|^2_\Omega \} \operatorname{vol}_\Omega \end{aligned}$$

Properties of $\mathcal{D}$:

- $\mathrm{Diff}_+(M)$-invariant
- positively homogenous ($\mathcal{D}(\lambda\Omega) = \lambda^{5/3}\mathcal{D}(\Omega)$ for $\lambda > 0$)
- Ω critical w.r.t. $\mathcal{D} \Leftrightarrow \Omega$ torsion-free

Let

$$Q(\Omega) := -\operatorname{grad} \mathcal{D}(\Omega)$$

be the negative L^2-gradient of $\mathcal{D}$.

The G_2-flow

Short-time existence

Consider the *G_2-flow* with initial condition $\Omega \in \Omega^3_+(M)$

$$\partial_t \Omega_t = Q(\Omega_t), \quad \Omega_0 = \Omega$$

for time-dependent family $\Omega_t \in \Omega^3_+(M)$, $t \geq 0$.

Theorem (W.-Witt)

The G_2-flow has a unique short-time solution.

Ingredients of proof:

- $\sigma_\xi(D_\Omega Q) \geq 0$ for all $\xi \in T^*M$
- $\ker \sigma_\xi(D_\Omega Q)$ precisely coming from diffeomorphism invariance
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The G_2-flow
Stability

The G_2-flow is stable near a critical point. More precisely:

Theorem (W.-Witt)

Let $\bar{\Omega} \in \Omega^3_+(M)$ be torsion-free. Then for any initial condition sufficiently close to $\bar{\Omega}$ in the C^∞-topology the G_2-flow exists for all times and converges modulo diffeomorphisms to a torsion-free positive 3-form on M.

Ingredients of proof:

- linear stability
- integrability of infinitesimal deformations
- compare nonlinear evolution with solution of linearized equation (estimates!)

The G_2-flow

Spinorial reformulation

Let Σ_n be the complex spin representation of Spin_n.

Representation theory: $\Sigma_7 = \Sigma_7^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, $\dim_{\mathbb{R}} \Sigma_7^{\mathbb{R}} = 8$.

Basic facts:

- Spin_7 acts transitively on $S(\Sigma_7^{\mathbb{R}}) \cong S^7$
- $S(\Sigma_7^{\mathbb{R}}) \cong \mathrm{Spin}_7/\mathrm{G}_2$
- $\Lambda^3_+ \mathbb{R}^{7*} \cong \mathrm{GL}_7^+/\mathrm{G}_2 \simeq \mathbb{R}P^7$

$1:1$ Correspondence

$$\Omega \longleftrightarrow \text{spin structure}, g, \{\pm\varphi\}$$

Then

$$\mathcal{D}(\Omega) = 8\int_M |\nabla^g \varphi|^2 \,\mathrm{vol}_g + \int_M \mathrm{scal}^g \,\mathrm{vol}_g$$