On the mass of asymptotically hyperbolic manifolds

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April 6, 2013,
EMS/DMF Joint Mathematical Weekend

Joint work with Romain Gicquaud and Anna Sakovich.

Motivation: asymptotically Euclidean case

Let $(\mathbb{R}^n, \delta)$ be Euclidean space. (M, g) is **asymptotically Euclidean** if

- There is a diffeomorphism $\Phi : M \setminus K \to \mathbb{R}^n \setminus \overline{B}_R(0)$,
- $e = \Phi_* g - \delta \to 0$ sufficiently fast at infinity.

$$m_{ADM} = \lim_{r \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \left(\operatorname{div}^\delta e - d(\operatorname{tr}^\delta e) \right) (\nu_r) \, d\mu^\delta .$$

Theorem (Schoen&Yau, Witten)

Suppose a complete AE manifold (M, g) has $\mathrm{Scal}^g \geq 0$. If either (M, g) has $\dim < 8$ or if (M, g) is spin then $m \geq 0$. Moreover, $m = 0$ iff (M, g) is isometric to $(\mathbb{R}^n, \delta)$.

Example. The slice $t = 0$ of the Schwarzschild spacetime is

$$\left(\mathbb{R}^n \setminus \{0\}, \left(1 + \frac{m}{2r^{n-2}} \right)^{\frac{4}{n-2}} \delta \right).$$

Asymptotically Euclidean, $\mathrm{Scal} = 0$ and $m_{ADM} = m$.

Asymptotically hyperbolic mass

Let $(\mathbb{H}^n, b)$ be the hyperbolic space. In polar coordinates $b = dr^2 + \sinh r\, \sigma$.

Roughly, (M, g) is **asymptotically hyperbolic** if

- There is a diffeomorphism $\Phi : M \setminus K \to \mathbb{H}^n \setminus \overline{B}_R(0)$,
- $e = \Phi_* g - b \to 0$ sufficiently fast at infinity.

Consider $\mathcal{N} = \{V \in C^\infty(\mathbb{H}^n) \mid \operatorname{Hess}^b V = Vb\}$. Vector space with a basis

$$V_{(0)} = \cosh r,\ V_{(1)} = x^1 \sinh r, \ldots,\ V_{(n)} = x^n \sinh r,$$

where $x^1, \ldots, x^n$ are the coordinate functions on $\mathbb{R}^n$ restricted to S^{n-1}.

The **mass functional** of (M, g) w.r.t. the chart Φ is

$$H_\Phi(V) = \lim_{r\to\infty} \int_{S_r} \Big(V(\operatorname{div}^b e - d \operatorname{tr}^b e) + (\operatorname{tr}^b e) dV - e(\nabla^b V, \cdot) \Big) (\nu_r)\, d\mu^b.$$

Asymptotically hyperbolic mass (continued)

Theorem (Wang; Chrusciel and Herzlich)

If (M, g) is a complete spin asymptotically hyperbolic manifold with $\mathrm{Scal}^g \geq -n(n-1)$ then H_Φ is either timelike future directed or zero. H_Φ is zero iff (M, g) is isometric to $(\mathbb{H}^n, b)$.

Let η be the Lorentzian inner product on $\mathcal{N}$ such that

- $\{V_{(0)}, V_{(1)}, \ldots, V_{(n)}\}$ is orthonormal with respect to η,
- $\eta(V_{(0)}, V_{(0)}) = -1$,
- $\eta(V_{(i)}, V_{(i)}) = 1$, $i = 1, \ldots, n$.

Then H_Φ is **timelike future directed** if $H_\Phi(V) > 0$ for every $0 < V \in \mathcal{N}_+$, where $\mathcal{N}_+$ is the interior of the future lightcone.
In this case the **mass** is defined as

$$m = \frac{1}{2(n-1)\omega_{n-1}} \inf_{\mathcal{N}^1} H_\Phi(V),$$

where $\mathcal{N}_1$ is the unit hyperboloid in $\mathcal{N}_+$.

Asymptotically hyperbolic mass (continued)

In fact, we can always find a chart Φ such that

$$m = \frac{1}{2(n-1)\omega_{n-1}} H_\Phi(V_{(0)}), \text{ where } V_{(0)} = \cosh r.$$

Such coordinates are called **balanced**.

Example. Let $\rho = \sinh r$. The slice $t = 0$ of the Anti-de Sitter-Schwarzschild spacetime outside the horizon has the metric

$$g_{\text{AdS-Schw}} = \frac{d\rho^2}{1 + \rho^2 - \frac{2m}{\rho^{n-2}}} + \rho^2 \sigma.$$

Asymptotically hyperbolic, $\mathrm{Scal} = -n(n-1)$, mass equals m.

Penrose conjecture for AE manifolds

There is a conjectured lower bound for the mass of AE manifolds in terms of their geometry:

Conjecture (Penrose 1973)

Let (M, g) be an n-dimensional AE manifold. If $\mathrm{Scal}^g \geq 0$ then

$$m_{ADM} \geq \frac{1}{2}\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}},$$

where Σ is the outermost minimal hypersurface. Equality holds iff (M, g) is isometric to the $t = 0$ slice of the Schwarzschild spacetime outside its minimal hypersurface.

- proven in $\dim = 3$ (Huisken and Ilmanen by IMCF; Bray 2001 Conformal flow);
- proof by Bray extended to $\dim < 8$ (Bray and Lee 2009);
- AE graphs (Lam 2010; Huang and Wu 2012).

Penrose conjecture for AH manifolds

Conjecture

Let (M, g) *be n-dimensional AH manifold. If* $\mathrm{Scal}^g \geq -n(n-1)$ *then*

$$m \geq \frac{1}{2}\left[\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} + \left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n}{n-1}}\right],$$

where Σ *is the outermost minimal hypersurface. Equality holds iff* (M, g) *is isometric to the* $t = 0$ *slice of the Anti-de Sitter-Schwarzschild spacetime outside its minimal hypersurface.*

- Neves 2010: IMCF is inconclusive in this case.
- Lee-Neves 2013: IMCF proof for cases of negative mass.
- Bray's conformal flow relies on the fact that on AE manifolds there are only two dimensional quantities m and $|\Sigma|$. On $\mathbb{H}^n$ there is a third one: curvature.

Penrose type inequalities for AH graphs

- $\mathbb{H}^{n+1} = (\mathbb{H}^n \times \mathbb{R}, b + V^2 ds \otimes ds)$, where $V = V_{(0)} = \cosh r$.
- Let $f : \mathbb{H}^n \setminus \Omega \to \mathbb{R}$ be a continuous function which is smooth on $\mathbb{H}^n \setminus \overline{\Omega}$, where Ω is relatively compact and open. Assume that $|df|(x) \to \infty$ when $x \to \partial\Omega$ and that $f_{|\partial\Omega} = \mathrm{const.}$
- Consider
$$M = \{(x, s) \in \mathbb{H}^n \times \mathbb{R} \mid f(x) = s\}$$
as a hypersurface in $\mathbb{H}^{n+1}$ with the induced metric.
- Consider $\Phi : M \to \mathbb{H}^n \setminus \Omega : (p, f(p)) \to p$. The pushforward of the induced metric is
$$g = b + V^2 df \otimes df.$$
- Let M be AH w.r.t. the chart Φ, i.e.
$$e = g - b = V^2 df \otimes df \to 0$$
sufficiently fast and let Φ be balanced.

$\Sigma = \partial M$ is a minimal surface with $|\Sigma| = |\partial\Omega|$.

Penrose type inequalities for AH graphs (continued)

Main observation:

$$V\left(\mathrm{Scal} + n(n-1)\right) = \operatorname{div}^b \left[\frac{V \operatorname{div}^b e - V d \operatorname{tr}^b e - e(\nabla V, \cdot) + (\operatorname{tr}^b e) dV}{1 + V^2 |df|^2} \right],$$

so that

$$H_\Phi(V) = \int_{\mathbb{H}^n \setminus \Omega} \frac{V[\mathrm{Scal} + n(n-1)]}{\sqrt{1 + V^2|df|^2}}\, d\mu^g + \int_{\partial\Omega} HV\, d\mu^b \geq \int_{\partial\Omega} HV\, d\mu^b.$$

Assuming that $B_{r_0}(0) \subset \Omega$ and that $H \geq 0$ we can estimate m from below.

- By Hoffman-Spruck inequality:

$$m \geq \frac{n-2}{2^n (n-1) n^{\frac{n}{n-1}}} V(r_0) \left(\frac{|\partial\Omega|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}},$$

- By Minkowski formula:

$$m \geq \frac{1}{2} V(r_0) \frac{|\partial\Omega|}{\omega_{n-1}}.$$

Penrose type inequalities for AH graphs (continued)

Theorem (de Lima, Girão, 2012)

Assume that Ω is star-shaped w.r.t. the origin, then the following inequality holds:

$$\int_{\Sigma} HV \, d\mu^b \geq \frac{1}{(n-1)\omega_{n-1}} \left[\left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} + \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n}{n-1}} \right],$$

with equality iff Ω is a round sphere centered at the origin.

- AH Penrose inequality holds for AH graphs (D.-Gicquaud-Sakovich-de Lima-Girão)
- Rigidity case from mean convex graph and maximum principles for scalar curvature (D.-Gicquaud-Sakovich following Huang-Wu for AE).

AE manifolds with small mass

The only AE manifold with zero mass is the Euclidean space.

Question: must an AE manifold (M, g) with $\mathrm{Scal}^g \geq 0$ and small mass be "close to" Euclidean space?

- Spinor methods (Finster et al): L^2-estimates for curvature tensor.
- Lee 2009: let (M, g) be **harmonically flat down to radius** R i.e.

$$\mathrm{Scal}^g = 0 \text{ and } g = U^{\frac{4}{n-2}}\delta \text{ on } \mathbb{R}^n \setminus \overline{B}_R.$$

  In this case: if $m \to 0$ then $g \to \delta$ uniformly on $\mathbb{R}^n \setminus \overline{B}_{\alpha R}$ for any $\alpha > 1$.

Lee's result used in Bray and Lee's proof of the Penrose conjecture.

AH manifolds with small mass

Consider the class $\mathcal{A}(R_0)$ of 4-tuples (M, g, Φ, U) such that

- (M, g) is AH w.r.t. $\Phi : M \setminus K \to \mathbb{H}^n \setminus \overline{B}_{R_0}$,
- $\mathrm{Scal}^g \geq -n(n-1)$ on M, and $\mathrm{Scal}^g = -n(n-1)$ on $M \setminus K$.
- $U > 0$, $\Phi_* g = U^{\frac{4}{n-2}} b$ on $\mathbb{H}^n \setminus B_{R_0}$, and $U \to 1$ at infinity,
- the coordinates Φ at infinity are balanced,
- the positive mass theorem holds for any asymptotically hyperbolic metric on M.

Theorem (D., Gicquaud, Sakovich 2012)

Let $R_1 > R_0$ and $\varepsilon > 0$. Then there is a constant $\delta > 0$ such that

$$|U - 1| \leq \varepsilon e^{-nr}$$

on $\mathbb{H}^n \setminus B_{R_1}$ for all $(M, g, \Phi, U) \in \mathcal{A}(R_0)$ with $m^g < \delta$.

AH manifolds with small mass (continued)

Idea of proof:

- Suppose that for a one parameter family of metrics $\{g_s\}$, $s \in [-s_0, s_0]$ the function $H(s) = H_\Phi^{g_s}(V_{(0)})$ is analytic with bounded coefficients. If $\dot{H}(0) > 0$ is too large w.r.t. small $H(0)$ we get a contradiction with the PMT.
- $\dot{H}(0)$ must be an integral of some geometric quantity. In AE case Lee uses $g_s = \lambda_s^{\frac{4}{n-2}}(g - s\chi\mathrm{Ric})$ where χ is a compactly supported function, λ_s is such that $\mathrm{Scal}^{g_s} = 0$. In this case $\dot{H}(0) = \int_M \chi|\mathrm{Ric}|^2 \, d\mu_g$.
- For AH the obvious try is $g_s = \lambda_s^{\frac{4}{n-2}}(g - s\chi(\mathrm{Ric} + (n-1)g))$ where χ is a compactly supported function, λ_s is such that $\mathrm{Scal}^{g_s} = -n(n-1)$. $\dot{H}(0)$ does not have a nice expression unless $\chi \equiv 1$.
- Reason: in the proof we use functions $V^g \approx V_{(0)}$ such that $\Delta^g V^g = nV^g$. They do not appear in the AE case.

AH manifolds with small mass (continued)

- Second try
$$g_s = \lambda_s^{\frac{4}{n-2}} \left(g - s\chi \left(\widehat{\mathrm{Ric}}^g - \frac{\mathring{\mathrm{Hess}}(V^g)}{V^g} \right) \right).$$
Then
$$\dot{H}(0) = \int_M \chi V^g \left| \widehat{\mathrm{Ric}}^g - \frac{\mathring{\mathrm{Hess}}(V^g)}{V^g} \right|_g^2 d\mu^g.$$
- Recall that small mass $H(0)$ implies small $\dot{H}(0)$.
- And if $\dot{H}(0) = 0$ then
$$\widehat{\mathrm{Ric}}^g = \frac{\mathring{\mathrm{Hess}}(V^g)}{V^g}$$
on $\mathbb{H}^n \setminus B_{R_1}$.
- In this case $-(V^g)^2 dt^2 + g$ is static. Kobayashi-Obata: U is the conformal factor for the Anti-de Sitter-Schwarzschild with $m = 0$ on $\mathbb{H}^n \setminus B_{R_1}$. Hence $U = 1$.

Thank you for your attention!