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Abstract

Spectral Theory

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On homogenization of periodic hyperbolic systems

The talk is devoted to homogenization of periodic differential operators. Let $B_\varepsilon = B_\varepsilon^* > 0$ be a second order matrix strongly elliptic differential operator acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. The coefficients of the operator B_ε depend on $\mathbf{x}/\varepsilon$, $0 < \varepsilon \leqslant 1$. Consider the hyperbolic system

$$\partial_t^2 \mathbf{u}_\varepsilon(\mathbf{x},t) = -B_\varepsilon \mathbf{u}_\varepsilon(\mathbf{x},t) + \mathbf{F}(\mathbf{x},t), \quad \mathbf{u}_\varepsilon(\mathbf{x},0) = 0, \quad (\partial_t \mathbf{u}_\varepsilon)(\mathbf{x},0) = \boldsymbol{\psi}(\mathbf{x}),$$

where $\boldsymbol{\psi} \in L_2(\mathbb{R}^d;\mathbb{C}^n)$ and $\mathbf{F} \in L_1((0,T);L_2(\mathbb{R}^d;\mathbb{C}^n))$ for some $0 < T \leqslant \infty$. Then

$$\mathbf{u}_\varepsilon(\cdot,t) = B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2})\boldsymbol{\psi} + \int_0^t B_\varepsilon^{-1/2}\sin((t-\tilde{t})B_\varepsilon^{1/2})\mathbf{F}(\cdot,\tilde{t})\,d\tilde{t}.$$

We are interested in the behaviour of the solution $\mathbf{u}_\varepsilon(\cdot,t)$ in the small period limit $\varepsilon \to 0$. It turns out that, for sufficiently smooth $\boldsymbol{\psi}$ and $\mathbf{F}$, the error estimates in approximations for the solution $\mathbf{u}_\varepsilon$ depend on suitable norms of $\boldsymbol{\psi}$ and $\mathbf{F}$ explicitly. In other words, we can approximate the operator $B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2})$ in a uniform operator topology:

$$\begin{aligned} &\|B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2}\sin(t(B^0)^{1/2})\|_{H^1(\mathbb{R}^d;\mathbb{C}^n)\to L_2(\mathbb{R}^d;\mathbb{C}^n)} \\ &\qquad \leqslant C\varepsilon|t|, \end{aligned} \tag{1}$$

$$\begin{aligned} &\|B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2}\sin(t(B^0)^{1/2}) - \varepsilon K_1(\varepsilon;t)\|_{H^2(\mathbb{R}^d;\mathbb{C}^n)\to H^1(\mathbb{R}^d;\mathbb{C}^n)} \\ &\qquad \leqslant C\varepsilon(1+|t|), \end{aligned} \tag{2}$$

$$\begin{aligned} &\|B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2}\sin(t(B^0)^{1/2}) - \varepsilon K_2(\varepsilon;t)\|_{H^3(\mathbb{R}^d;\mathbb{C}^n)\to L_2(\mathbb{R}^d;\mathbb{C}^n)} \\ &\qquad \leqslant C\varepsilon^2(1+t^2). \end{aligned} \tag{3}$$

Here B^0 is the so-called effective operator with constant coefficients, $K_1(\varepsilon;t)$ and $K_2(\varepsilon;t)$ are the correctors. The correctors contain rapidly oscillating factors and so depend on ε.

We derive estimates (1) and (2) from the corresponding approximations for the resolvent B_ε^{-1}, obtained by T. A. Suslina (2010) with the help of the spectral

theory approach to homogenization problems. Our method is a modification of the classical proof of the Trotter-Kato theorem. The analogue of estimate (3) is also known (Suslina, 2014). But the author have no idea how to modify the Trotter-Kato theorem for this case. So, to prove estimate (3), we directly apply the spectral theory approach in a version developed by M. Sh. Birman and T. A. Suslina. The technique is based on the unitary scaling transformation, the Floquet-Bloch theory, and the analytic perturbation theory.