$\mathbb{C}P^1$-structures and dynamics in moduli space

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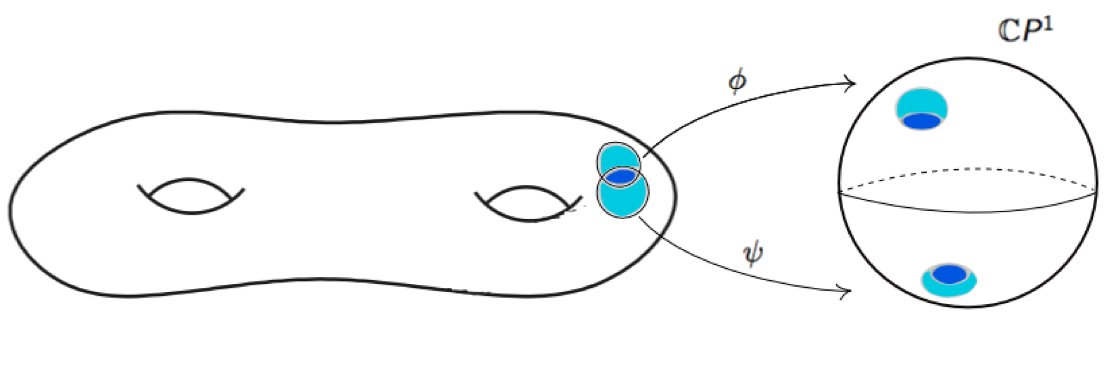
What is a $\mathbb{C}P^1$-structure?

A *complex projective structure* on a surface S is a choice of charts to $\mathbb{C}P^1$ such that the transition maps lie in $PSL(2,\mathbb{C})$. It has a **developing map** $f: \widetilde{S} \to \mathbb{C}P^1$ and a **holonomy representation** $\rho: \pi_1(S) \to PSL(2,\mathbb{C})$ that records the mismatch of charts around loops.



$\phi \circ \psi^{-1}$ is a Möbius map

Example: A **hyperbolic surface**, with charts to the hyperbolic plane $\mathbb{H}^2$ thought of as the upper hemisphere of $\mathbb{C}P^1$, and transition maps in $Isom^+(\mathbb{H}^2) = PSL_2(\mathbb{R}) \hookrightarrow PSL_2(\mathbb{C})$. The holonomy is a **Fuchsian representation**.

The bundle picture

Assume S is closed, oriented and of genus $g \geq 2$.

Let $\mathcal{P}_g = \{\text{space of projective structures on } S\}$.

Since the transition maps above are conformal, they also define a complex structure on S. So we have:

$$\begin{array}{ccc} \mathcal{P}_g & \xrightarrow{h} & \chi(S) \\ \downarrow & & \\ \mathcal{T}_g & & \\ \downarrow & & \\ \mathcal{M}_g & & \end{array}$$

Here $\mathcal{T}_g$ is the **Teichmüller space** of all "marked" Riemann surfaces, and $\mathcal{M}_g$ is **Riemann's moduli space**, the quotient by the action of the *mapping class group.*

$\chi(S)$ is the **character variety** of $PSL(2,\mathbb{C})$-representations of $\pi_1(S)$ upto conjugation, and h maps a $\mathbb{C}P^1$-structure to its holonomy.

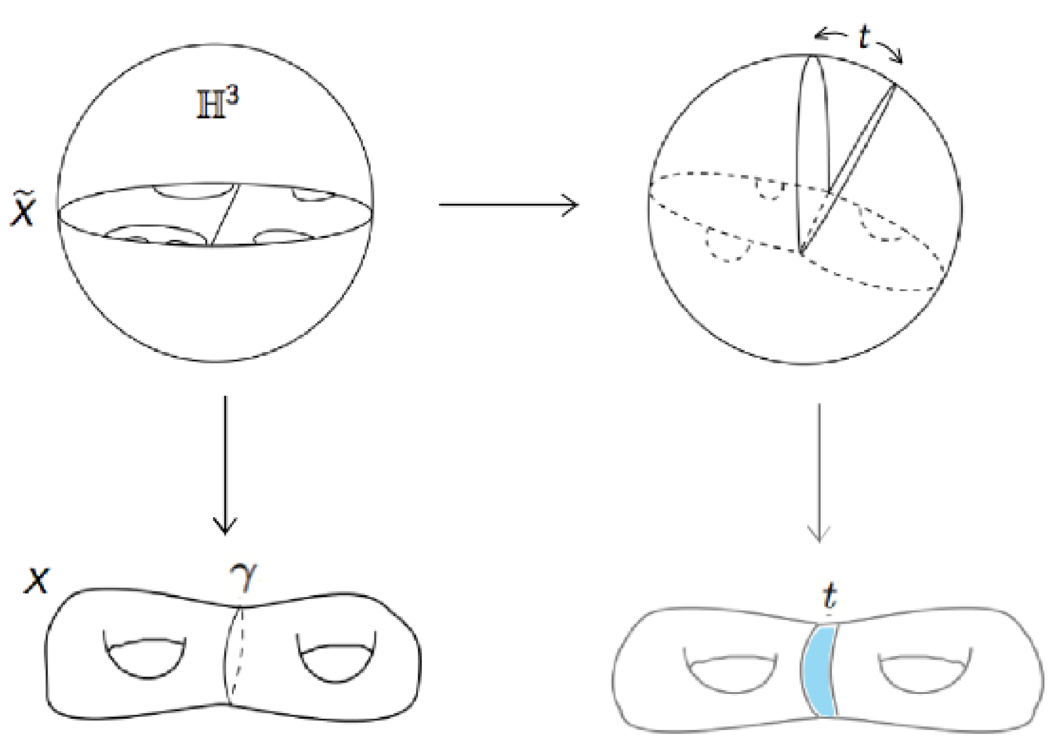
A motivating question

Given a representation $\rho \in \chi(S)$, what is the structure of the level set $h^{-1}(\rho)$?

In 1983 Faltings conjectured that for ρ Fuchsian this set is infinite - this was shown to be true in [2]. More recently, work of Shinpei Baba shows that a generic level set is infinite.

Bending and grafting

Given a hyperbolic surface X and a simple closed geodesic γ , one can deform the $\mathbb{C}P^1$-structure by *bending* the developing map.



This amounts to grafting in a euclidean cylinder at γ. More generally, one can graft along a **lamination** λ, which is a *limit* of curves.

The "bending angle" gives a one-parameter family of deformations

$$\{gr_{t\lambda}X\}_{t\geq 0}$$

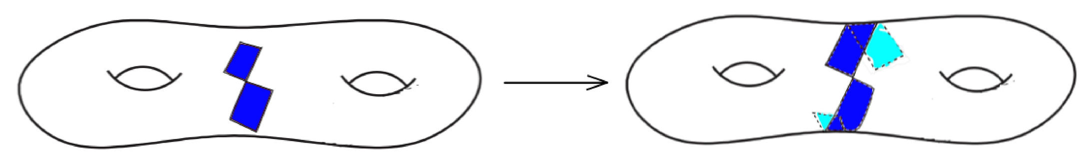
whose projection to $\mathcal{T}_g$ is a **grafting ray**.

Teichmüller geodesic rays

For two Riemann surfaces X and Y one can construct **quasiconformal maps** between them that allow a bounded distortion of angle.

For each such f, this distortion is some quantity K and the **Teichmüller distance** is given by the *least* distortion map:

$$d_{\mathcal{T}}(X,Y) = \frac{1}{2}\inf_f \ln K$$



Such a map picks a decomposition of the surface into rectangles, and stretches horizontally along each *(the figure above gives a partial picture).*

Given X and a horizontal foliation λ, the *stretch factor* parametrizes a geodesic path

$$\{Teich_{t\lambda}X\}_{t\geq 0}$$

in $\mathcal{T}_g$ which is a **Teichmüller ray**.

Our results

Theorem 1. Let $X \in \mathcal{T}_g$ and λ any lamination. Then there exists a $Y \in \mathcal{T}_g$ such that the grafting ray determined by (X,λ) is **strongly asymptotic** to the Teichmüller ray determined by (Y,λ), that is,

$$d_{\mathcal{T}}(gr_{e^t\lambda}X, Teich_{t\lambda}Y) \to 0$$

as $t \to \infty$.

By the *ergodicity* of the Teichmüller geodesic flow (see [6]), we have:

Corollary. Almost every grafting ray projects to a dense set in $\mathcal{M}_g$.

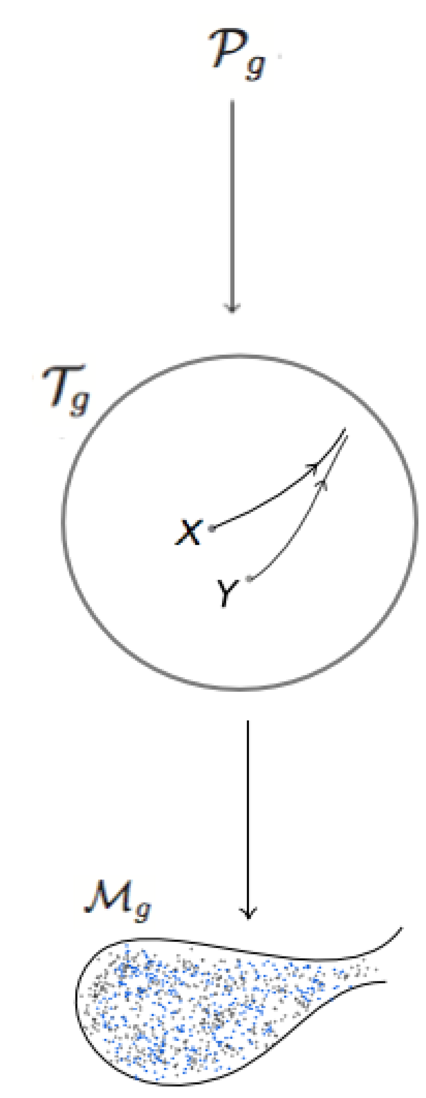
Theorem 2. Let $X \in \mathcal{T}_g$. Then the set

$$\mathcal{S} = \{gr_{2\pi\gamma}X \mid \gamma \text{ is a multicurve }\}$$

projects to a **dense** set in moduli space $\mathcal{M}_g$.

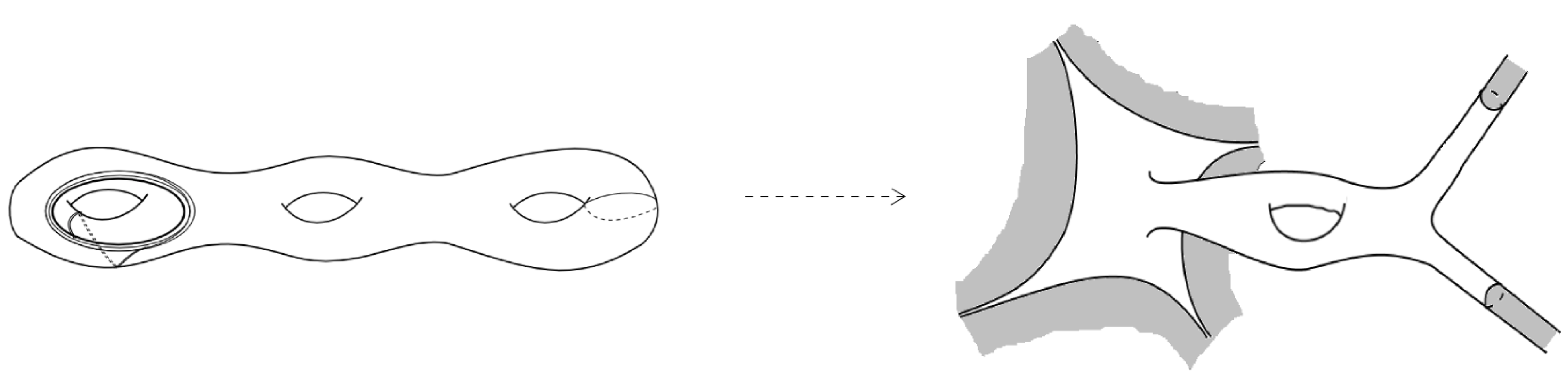
Since these "2π-graftings" preserve Fuchsian holonomy we get:

Corollary. For ρ a Fuchsian representation, $h^{-1}(\rho)$ projects to a dense set in $\mathcal{M}_g$.



Idea of the proof of the asymptoticity result

- Take a **conformal limit** of the grafting ray as $t \to \infty$:



The "infinitely-grafted surface" is obtained by gluing in euclidean half-planes and half-infinite cylinders to the complement of the lamination.

- "Uniformize" this to an infinite-area *singular flat surface* by prescribing a **meromorphic quadratic differential with higher order poles**. This shall be the "limit" of the asymptotic Teichmüller ray.
- Use **quasiconformal cut-and-paste** to adjust this uniformizing map to an *almost-conformal* map between surfaces along the rays.

References

[1] D. Dumas, *Complex projective structures.* In "Handbook of Teichmüller theory" Vol. II, EMS Zürich, 2004.

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[3] S. Gupta, *Asymptoticity of grafting and Teichmüller rays I.* http://arxiv.org/abs/1109.5365

[4] –, *Meromorphic quadratic differentials with half-plane structures.* http://arxiv.org/abs/1301.0332

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[6] H. Masur, *Interval exchange transformations and measured foliations.* Ann. of Math. (2), Vol. 115 No.1, 1982.