The Stochastics of Energy Markets
...or...
Modelling Financial Energy Forwards

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Overview

- Goal: Model the forward price dynamics in power markets
- Why?
  - Price and hedge options and other derivatives
  - Risk management (hedge production and price risk)

1. Some stylized facts of energy forward prices
2. Levy processes in Hilbert space
   - Subordination of Wiener processes
3. Modelling the forward dynamics
   - Adopting the Heath-Jarrow-Morton (HJM) dynamical modelling from interest rate theory
4. Ambit fields and forward prices
   - A direct HJM approach

1. Forward markets

Energy forward contracts

- Forward contract: a promise to deliver a commodity at a specific *future* time in return of an agreed price
  - Examples: coffee, gold, oil, orange juice, corn....
  - or.... temperature, rain, electricity
- Electricity: future delivery of power over a period in time
  - A given week, month, quarter or year
- The agreed price is called the *forward price*
  - Denominated in Euro per MWh
  - Forward contracts traded at EEX, NordPool, etc...
  - Financial delivery!

- Forward price at time $t \leq T_1$, for contract delivering over $[T_1, T_2]$, denoted by $F(t, T_1, T_2)$
- Connection to fixed-delivery forwards

$$F(t, T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} f(t, T) \, dT$$

- Musiela parametrization: $x = T_1 - t, y = T_2 - T_1$

$$G(t, x, y) = F(t, t + x, t + x + y) \,, \quad g(t, x) = f(t, t + x)$$

- Focus on modelling the dynamics of the *forward curve*

$$t \mapsto g(t, x)$$

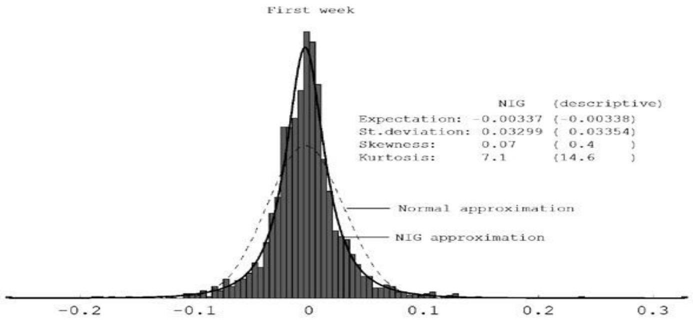
Some stylized facts of power forwards

- Consider the *logreturns* from observed forward prices (at NordPool)

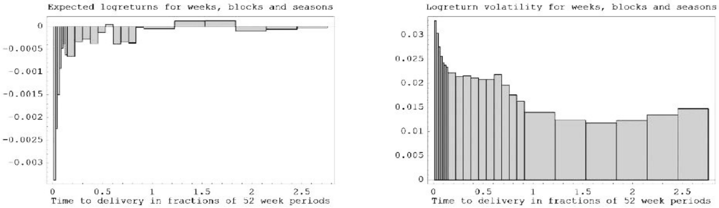
$$r_i(t) = \ln \frac{F(t, T_{1i}, T_{2i})}{F(t-1, T_{1i}, T_{2i})}$$

- General findings are:
  1. Distinct heavy tails across all segments
  2. No significant skewness
  3. Volatilities (stdev's) are in general falling with time to delivery $x = T_1 - t$ (Samuelson effect)
  4. Significant correlation between different maturities x (idiosyncratic risk)

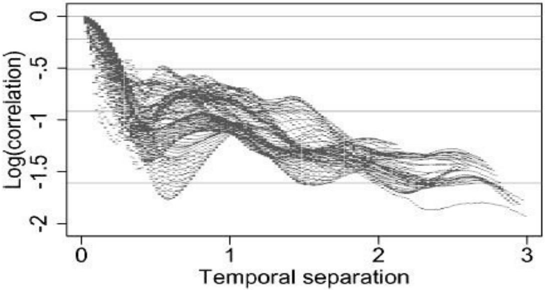
- Fitting NIG and normal to logreturns of forwards by maximum likelihood



- Expected logreturn (left) and volatility (right)



- Plot of log-correlation as a function of years between delivery
- Correlation decreases in general with distance between delivery
  - ...but in a highly complex way



Summary of empirical evidence

- Forward curve $g(t,x)$ is a random field in time and space
  - Or, a stochastic process with values in a function space
- Strong dependencies between maturity times x
  - High degree of idiosyncratic risk in the market
- Non-Gaussian distributed log-returns
  - Dynamics is not driven by Brownian motion

2. Hilbert space-valued Lévy processes

- Goal: construct a Hilbert-space valued Lévy process with given characteristics
  - For example, a normal inverse Gaussian (NIG) Lévy process in Hilbert space
- X is a d-dimensional NIG random variable if

$$X\Big|\sigma^2 \sim \mathcal{N}_d(\mu + \beta\sigma^2, \sigma^2 C)$$

- $\mu \in \mathbb{R}^d$, $\beta \in \mathbb{R}$, C $d \times d$ covariance matrix,
- σ an inverse Gaussian random variable
- X defined by a mean-variance mixture model

Lévy processes by subordination

- Define a NIG Lévy process $L(t)$ with values in Hilbert space by subordination
- In general: let
  - H be a separable Hilbert space
  - Θ a real-valued subordinator, that is, a Lévy process with increasing paths
  - W a drifted H-valued Brownian motion with covariance operator Q and drift b
  - Q is symmetric, positive definite, trace-class operator,

$$\mathrm{Cov}(W)(f,g) = \mathbb{E}\left[\langle W(1)-b, f\rangle\langle W(1)-b, g\rangle\right] = \langle Qf, g\rangle$$

- Define

$$L(t) = W(\Theta(t))$$

- Let ψ_Θ be the cumulant (log-characteristic) function of Θ
- Cumulant of L becomes

$$\psi_L(z) = \psi_\Theta\left(\mathrm{i}\langle z, b\rangle - \frac{1}{2}\langle Qz, z\rangle\right), z \in H$$

- Let $(a, 0, \ell)$ be characteristic triplet of Θ, then triplet of L is (β, aQ, ν)

$$\beta = ab + \int_0^\infty \mathbb{E}[\mathbf{1}(|W(t)| \leq 1)]\,\ell(dz)$$

$$\nu(A) = \int_0^\infty P^{W(t)}(A)\,\ell(dt), A \subset H, \text{Borel}$$

- Suppose L square-integrable Lévy process
- Define covariance operator

$$\mathrm{Cov}(L)(f,g) = \mathbb{E}\left[\langle L(1), f\rangle\langle L(1), g\rangle\right] = \langle \mathcal{Q}f, g\rangle$$

  - Supposing mean-zero Lévy process
  - $\mathcal{Q}$ symmetric, positive definite, trace-class operator
- If L is defined via subordination, covariance operator is

$$\mathcal{Q} = \mathbb{E}[\Theta(1)]Q$$

  - Supposing $\Theta(1)$ integrable

- So, how to obtain L being NIG Lévy process?
- Choose Θ to be driftless inverse Gaussian Lévy process, with Lévy measure

$$\ell(dz) = \frac{\gamma}{2\pi z^3} \mathrm{e}^{-\delta^2 z/2} \mathbf{1}(z > 0)\, dz$$

- Define $L(t) = W(\Theta(t))$, which we call a H-valued NIG Lévy process with triplet $(\beta, 0, \nu)$,

Theorem

L is a H-valued NIG Lévy process if and only if $TL(t)$ is a $\mathbb{R}^n$-valued NIG Lévy process for every linear operator $T : H \mapsto \mathbb{R}^n$.

3. Forward price dynamics

- Let H be a separable Hilbert space of real-valued continuous functions on $\mathbb{R}_+$
  - with δ_x, the evaluation map, being continuous
  - $x \in \mathbb{R}_+$ is *time-to-maturity*
  - H is, e.g. the space of all absolutely continuous functions with derivative being square integrable with respect to an exponentially increasing function (Filipovic 2001)
- Assume L is square-integrable zero-mean Lévy process
  - Defined on a separable Hilbert space U, typically being a function space as well (e.g. $U = H$)
  - Triplet (β, Q, ν) and covariance operator $\mathcal{Q}$

- Define process X on H as the solution of

$$dX(t) = (AX(t) + a(t))\,dt + \sigma(t)\,dL(t)$$

- $A = d/dx$, generator of the C_0-semigroup of shift operators on H
- $a(\cdot)$ H-valued process, $\sigma(\cdot)$ $L_{HS}(\mathcal{H}, H)$-valued process being predictable
  - $L_{HS}(\mathcal{H}, H)$, space of Hilbert-Schmidt operators, $\mathcal{H} = \mathcal{Q}^{1/2}(U)$

$$\mathbb{E}\left[\int_0^t \|\sigma(s)\mathcal{Q}^{1/2}\|^2_{L_{\mathrm{HS}}(U,H)}\,ds\right] < \infty$$

- σ and a may be functions on the state again
  - We will not assume that generality here

- Mild solution, with S as shift operator

$$X(t) = S(t)X_0 + \int_0^t S(t-s)a(s)\,ds + \int_0^t S(t-s)\sigma(s)\,dL(s)$$

- Define forward price $g(t,x)$ by

$$g(t,x) = \exp(\delta_x(X(t)))$$

- By letting $x = T - t$, we reach the actual forward price dynamics

$$f(t,T) = g(t, T-t)$$

- Assume X is modelled under "risk-neutrality", then $f(\cdot, T)$ must be a martingale
    - Yields conditions on a and σ!
- Introduce

$$\widehat{a}(t) = \int_0^t a(s)(T-s)\,ds\,, \quad \widehat{\sigma}(t) = \int_0^t \delta_0 S(T-s)\sigma(s)\,dL(s)$$

Theorem

The process $t \mapsto f(t,T)$ for $t \leq T$ is a martingale if and only if

$$d\widehat{a}(t) = -\frac{1}{2}d[\widehat{\sigma},\widehat{\sigma}]^c(t) - \{e^{\Delta\widehat{\sigma}(t)} - 1 - \Delta\widehat{\sigma}(t)\}$$

- $\Delta\widehat{\sigma}(t) = \widehat{\sigma}(t) - \widehat{\sigma}(t-)$, $[\widehat{\sigma},\widehat{\sigma}]^c$ continuous part of bracket process of $\widehat{\sigma}$

Market dynamics

- Forward model under risk neutral probability $\mathbb{Q}$
- Esscher transform $\mathbb{Q}$ to "market probability" $\mathbb{P}$ to get market dynamics of F
- Let $\phi(\theta)$ be the log-moment generating function (MGF) og L
  - Recall characteristic triplet of L as (β, Q, ν)
  - Assume L is exponentially integrable

$$\begin{aligned}\phi(\theta) &= \ln \mathbb{E}[\mathrm{e}^{(\theta, L(1))_U}] \\ &= (\beta, \theta)_U + \frac{1}{2}(Q\theta, \theta)_U \\ &\qquad + \int_U \mathrm{e}^{(\theta, y)_U} - 1 - (\theta, y)_U \mathbf{1}_{|y|_U \leq 1}\, \nu(dy)\,, \theta \in U\end{aligned}$$

- $d\mathbb{P}/d\mathbb{Q}$ conditioned on $\mathcal{F}_t$ has density

$$Z(t) = \exp\left((\theta, L(t))_U - \phi(\theta)\, t\right)$$

- Lévy property of L preserved under Esscher transform
- Characteristic triplet under $\mathbb{P}$ is $(\beta_\theta, Q, \nu_\theta)$

$$\beta_\theta = \beta + \int_{|y|_U \leq 1} y\, \nu_\theta(dy)\,, \qquad \nu_\theta(dy) = \mathrm{e}^{(\theta, y)_U}\, \nu(dy)$$

- $\theta \in U$ is the *market price of risk*
  - Esscher transform will shift the drift in X-dynamics, and
  - and rescale (exponentially tilt) the jumps of L

Example

- $L = W$, Wiener process in U
- Bracket process can be computed to be

$$[\widehat{\sigma}, \widehat{\sigma}]^c(t) = \int_0^t \|\delta_0 S(T-s)\sigma(s)Q^{1/2}\|^2_{L_{HS}(U,\mathbb{R})}\, ds$$

- An example by Audet et al. (2004)
- Volatility specification
  - σ multiplication operator: $\delta_x \sigma(t) u = \eta \mathrm{e}^{-\alpha x} u(x)$, $u \in U$
  - η, α positive constants, α mean-reversion speed
  - Volatility structure linked to an exponential Ornstein-Uhlenbeck process for the spot

- Spatial covariance structure of W
  - Let Q be integral operator
  - $q(x,y)=\exp(-\kappa|x-y|)$ integral kernel
- Recall correlation structure from empirical studies.....
  - ...close to exponentially decaying
  - Some seasonal variations: let η be seasonal
- Forward dynamics of Audet et al. (2004)

$$\ln\frac{g(t,x)}{g(0,x)}=-\frac{1}{2}\eta^2\int_0^t \mathrm{e}^{-2\alpha(x+t-s)}\,ds+\int_0^t \eta\mathrm{e}^{-\alpha(x+t-s)}\,dW(s,x)$$

- Or....

$$\frac{df(t,T)}{f(t,T)}=\eta\mathrm{e}^{-\alpha(T-t)}\,dW(t,T-t)$$

- Note: series representation of W
  - Independent Gaussian processes, $\{e_n\}$ basis of U

$$W(t) = \sum_{n=1}^{\infty} \langle W(t), e_n \rangle_U e_n$$

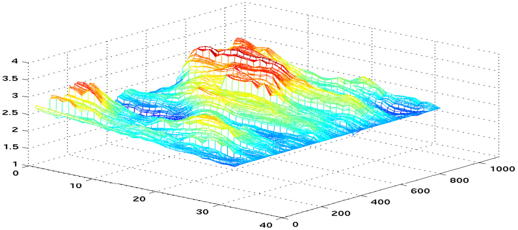
- May represent the dynamics in terms of Brownian factors
  - Infinite factor model
- Recall the heavy tails in log-return data for NordPool forwards
  - A Wiener specification W is not justified
- Should use an exponential NIG-Lévy dynamics instead
  - Choose L to be NIG, constructed by subordinator
  - Keep covariance operator

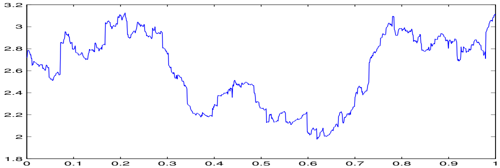
Numerical examples with NIG-Levy field

- Simulation of forward field by numerically solving the hyperbolic stochastic partial differential equation for X
  - Euler discretization in time
  - A finite-element method in "space" x
  - Conditions at "inflow" boundary "$x = \infty$" and at $t = 0$
- Initial condition $X(0, x)$ is "today's observed forward curve" on log-scale
  - Exponentially decaying curve
  - Motivated from "typical" market shapes
- Boundary condition at infinity equal to constant
  - Stationary spot price dynamics yield a constant forward price at "infinite maturity"

- L is supposed to be a NIG-Lévy process, which is defined as a subordination
- Appeal to the series expansion of W, which is truncated in the numerics
  - Simulate a path of an inverse Gaussian Lévy process
  - Change time of the finite set of independent Brownian motions
  - Sum up these scaled by eigenvalues and basis function to get the NIG-Lévy field approximation
- Parameters
  - $\alpha = 0.2$, mean-reversion
  - $\kappa = 2$, correlation
  - IG-parameters chosen by convenience $(\gamma = 10, \delta = 1)$

- Forward field, for $x = 0, .., 40$ days to maturity, and t daily over 4 years. Implied spot process for $x = 0$





- Can we recover the spot dynamics from the forward model?

Implied spot price dynamics

- One can recover the spot dynamics as

$$g(t,0) = \exp(\delta_0(X(t)))$$

- X is driven by by NIG Lévy process in U
    - "Infinitely" many Lévy processes
- For $\widetilde{L}$ is univariate NIG Lévy process, $\widetilde{\sigma}$ stochastic process on $\mathbb{R}$, it holds

$$\delta_0 \int_0^t \sigma(s)\, dL(s) = \int_0^t \widetilde{\sigma}(s)\, d\widetilde{L}(s)$$

- Spot can be represented as a dynamics in terms of a univariate NIG Lévy process

4. HJM modeling by ambit fields

Forward dynamics by ambit fields

- A twist on the HJM approach
    - by direct modelling rather than as the solution of some dynamic equation
    - Barndorff-Nielsen, B., Veraart (2010b)
- Simple *arithmetic* model in the risk-neutral setting

$$g(t,x)=\int_{-\infty}^{t}\int_{0}^{\infty}k(t-s,x,y)\sigma(s,y)L(dy,ds)$$

- L is a *Lévy basis*, k non-negative deterministic function, $k(u,x,y)=0$ for $u<0$, stochastic volatility process σ (typically independent of L and stationary)

- L is a *Lévy basis* on $\mathbb{R}^d$ if
  1. the law of $L(A)$ is infinitely divisible for all bounded sets A
  2. if $A \cap B = \emptyset$, then $L(A)$ and $L(B)$ are independent
  3. if $A_1, A_2, \ldots$ are disjoint bounded sets, then

$$L(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} L(A_i)\,, a.s$$

- Stochastic integration in time and space: use the Walsh-definition (for *square integrable* Lévy bases)
  - Natural adaptedness condition on σ
  - square integrability on $k(t-\cdot, x, \cdot) \times \sigma$ with respect to covariance operator of L
- Possible to relate ambit fields to Hilbert-space valued processes

Martingale condition

- No-arbitrage conditions: $t \mapsto f(t, T) := g(t, T_t)$ must be a martingale

Theorem

$f(t, T)$ is a martingale if and only if there exists $\widetilde{k}$ such that

$$k(t - s, T - t, y) = \widetilde{k}(s, T, y)$$

- Note, cancellation effect on t in 1st and 2nd argument ensures martingale property

- Example 1: exponential damping function (motivated by OU spot models)

$$k(u, x, y) = \exp\left(-\alpha(u + x + y)\right)$$

- Satisfies the martingale condition

$$k(t - s, T - t, y) = \exp\left(-\alpha(y + T - s)\right) =: \widetilde{k}(s, T, y)$$

- Example 2: the SPDE specification of f
    - Let $L = W$, a univariate Brownian motion for simplicity

$$dg(t, x) = \frac{\partial g}{\partial x}(t, x)\, dt + \sigma(t, x)\, dW(t)$$

- Solution of the SPDE

$$g(t,x) = g_0(x+t) + \int_0^t \sigma(s, x+(t-s))\, dW(s)$$

- Note: forward price $g(t,x)$ is an ambit process
- Letting $x = T - t$,

$$g(t, T-t) = g_0(T) + \int_0^t \sigma(s, T-s)\, dW(s)$$

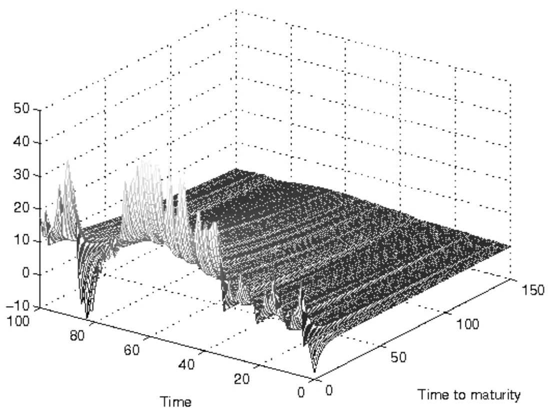
- Martingale condition is satisfied....of course!

Example

- Suppose k is a weighted sum of two exponentials
  - Motivated by a study of spot prices on the German EEX
  - ARMA(2,1) in continuous time

$$k(t-s,x,y) = w\exp(-\alpha_1(t-s+x+y)) + (1-w)\exp(-\alpha_2(t-s+x-y))$$

- $L = W$ a Gaussian basis
- $\sigma(s,y)$ again an ambit field
  - Exponential kernel function
  - Driven by inverse Gaussian Lévy basis



- Spot is very volatile
- Rapid convergence to zero when time to maturity increases
    - In reality there will be a seasonal level

Thank you for your attention!

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