Edge state integrals on shaped triangulations

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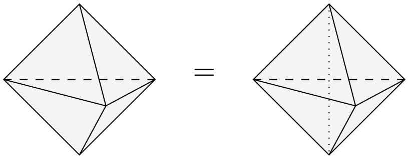
Motivation: quantum Chern–Simons theory with a non-compact gauge group

Given a Lie group G, a 3-manifold M. Chern–Simons action functional $CS_M(A) = \int_M \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$.
Gauge fields: G-connections $A \in \mathcal{A} = \Omega^1(M, Lie\ G)$.
Group of gauge transformations $\mathcal{G} = \mathcal{C}^\infty(M, G)$,

$$\mathcal{A} \times \mathcal{G} \to \mathcal{A}, \quad (A, g) \mapsto A^g := g^{-1} A g + g^{-1} dg$$

Phase space $=$ space of flat connections $= \hom(\pi_1(M), G)/G$.
Partition function: $Z_\hbar(M) = \int_{\mathcal{A}/\mathcal{G}} e^{\frac{i}{\hbar} CS_M(A)} \mathcal{D}A$.
Problem: give a mathematically rigorous definition of this partition function.
Previous works: Witten, Hikami, Dijkgraaf, Fuji, Manabe, Dimofte, Gukov, Lenells, Zagier, Gaiotto, Andersen, K.

Topological invariance and the $2-3$ Pachner move



The Ponzano–Regge model of $2+1$-dimensional quantum gravity: states on edges (finite-dimensional representations of $sl(2)$) and weights on tetrahedra ($6j$-symbols).
The Turaev–Viro model: replace $sl(2)$ by $U_q(sl(2))$ and fix q by a root of unity.
Next steps: infinite-dimensional representations, generic q's.
Need for a special function.

Faddeev's quantum dilogarithm

For $\hbar \in \mathbb{R}_{>0}$, Faddeev's quantum dilogarithm function is defined by

$$\Phi_\hbar(z) = \exp\left(\int_{\mathbb{R}+i\epsilon} \frac{e^{-i2xz}}{4\sinh(xb)\sinh(xb^{-1})x} dx\right)$$

in the strip $|\Im z| < \frac{1}{2\sqrt{\hbar}}$, where $\hbar = (b + b^{-1})^{-2}$, and extended to the whole complex plane through the functional equations

$$\Phi_\hbar(z - ib^{\pm 1}/2) = (1 + e^{2\pi b^{\pm 1} z})\Phi_\hbar(z + ib^{\pm 1}/2)$$

One can choose $\Re b > 0$ and $\Im b \geq 0$. If $\Im b > 0$ (i.e. $\hbar > 1/4$), then one can show that

$$\Phi_\hbar(z) = \frac{(-qe^{2\pi bz}; q^2)_\infty}{(-\bar{q}e^{2\pi b^{-1} z}; \bar{q}^2)_\infty}$$

where $q := e^{i\pi b^2}$, $\bar{q} := e^{-i\pi b^{-2}}$, and

$$(x; y)_\infty := (1 - x)(1 - xy)(1 - xy^2)\dots$$

Analytical properties of Faddeev's quantum dilogarithm

Zeros and poles:

$$(\Phi_\hbar(z))^{\pm 1} = 0 \;\Leftrightarrow\; z = \mp\left(\frac{i}{2\sqrt{\hbar}} + mib + nib^{-1}\right),\;\; m, n \in \mathbb{Z}_{\geq 0}$$

Behavior at infinity:

$$\Phi_\hbar(z)\Big|_{|z|\to\infty} \approx \begin{cases} 1 & |\arg z| > \frac{\pi}{2} + \arg b \\ \zeta_{inv}^{-1} e^{i\pi z^2} & |\arg z| < \frac{\pi}{2} - \arg b \\ \frac{(\bar{q}^2;\bar{q}^2)_\infty}{\Theta(ib^{-1}z;-b^{-2})} & |\arg z - \frac{\pi}{2}| < \arg b \\ \frac{\Theta(ibz;b^2)}{(q^2;q^2)_\infty} & |\arg z + \frac{\pi}{2}| < \arg b \end{cases}$$

where $\zeta_{inv} := e^{\pi i(2+\hbar^{-1})/12}$, $\Theta(z;\tau) := \sum_{n\in\mathbb{Z}} e^{\pi i\tau n^2 + 2\pi i z n}$, $\Im\tau > 0$.
Inversion relation:

$$\Phi_\hbar(z)\Phi_\hbar(-z) = \zeta_{inv}^{-1} e^{i\pi z^2}.$$

Complex conjugation:

$$\overline{\Phi_\hbar(z)}\Phi_\hbar(\bar{z}) = 1.$$

Quantum five term identity

Heisenberg's (normalized) selfadjoint operators in $L^2(\mathbb{R})$

$$\mathbf{p}f(x) := \frac{1}{2\pi i}f'(x), \quad \mathbf{q}f(x) := xf(x)$$

Quantum five term identity for unitary operators

$$\Phi_\hbar(\mathbf{p})\Phi_\hbar(\mathbf{q}) = \Phi_\hbar(\mathbf{q})\Phi_\hbar(\mathbf{p}+\mathbf{q})\Phi_\hbar(\mathbf{p})$$

Equivalent integral formula

$$\int_{\mathbb{R}} \frac{\Phi_\hbar(x+u)}{\Phi_\hbar\left(x - \frac{i}{2\sqrt{\hbar}} + i0\right)} e^{-2\pi i w x}\, dx = \zeta_o \frac{\Phi_\hbar(u)\,\Phi_\hbar\left(\frac{i}{2\sqrt{\hbar}} - w\right)}{\Phi_\hbar(u-w)}$$

where $\zeta_o := \exp\left(\frac{\pi i}{12}\left(1+\frac{1}{\hbar}\right)\right)$, and $0 < \Im w < \Im u < \frac{1}{2\sqrt{\hbar}}$.
In particular,

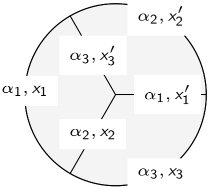
$$\int_{\mathbb{R}+i\epsilon} \Phi_\hbar(x) e^{-2\pi i w x}\, dx = \zeta_o e^{-\pi i w^2} \Phi_\hbar\left(\frac{i}{2\sqrt{\hbar}} - w\right)$$

Notation for CW-complexes:

- $\Delta_i(X)$ = the set of i-dimensional simplices of X
- $\Delta_{i,j}(X) = \{(a,b)|\ a \in \Delta_i(X),\ \ b \in \Delta_j(a)\}$

Two types of edge labelings:

- State variables $x\colon \Delta_1(X) \to \mathbb{R}$;
- Shape variables $\alpha\colon \Delta_{3,1}(X) \to]0,\pi[$, $\alpha(t,e) = \alpha(t,e^{\mathrm{op}})$, $\sum_e \alpha(t,e) = 2\pi$.



$$\alpha_1 + \alpha_2 + \alpha_3 = \pi$$

Neumann–Zagier symplectic structure: $\omega_{NZ} = d\alpha_1 \wedge d\alpha_2$

The tetrahedral weight function

The weight function of a tetrahedron T in state x and with shape α:

$$W_\hbar(T,x,\alpha)=\prod_{j=1}^{3}\Psi_\hbar\left(x_{j+1}+x'_{j+1}-x_{j-1}-x'_{j-1}+\frac{i}{\sqrt{\hbar}}\left(\frac{1}{2}-\frac{\alpha_j}{\pi}\right)\right)$$

where

$$\Psi_\hbar(x)=\frac{\Phi_\hbar(x)}{\Phi_\hbar(0)}e^{-i\pi x^2/2},\quad \Psi_\hbar(x)\Psi_\hbar(-x)=1$$

The weight function of a triangulation X in state x and with shape α:

$$W_\hbar(X,x,\alpha)=\prod_{T\in\Delta_3(X)}W_\hbar(T,x,\alpha)$$

The partition function

Denote

$$\mathbb{R}^{\Delta_j(X)} = \{f : \Delta_j(X) \to \mathbb{R}\}, \quad j \in \{0, 1\}.$$

State gauge transformations

$$\mathbb{R}^{\Delta_1(X)} \times \mathbb{R}^{\Delta_0(X)} \to \mathbb{R}^{\Delta_1(X)}, \quad (x, g) \mapsto x^g,$$

$$x^g(e) = x(e) + g(v_1) + g(v_2), \quad \partial e = \{v_1, v_2\}.$$

State gauge invariance of the weight function:

$$W_\hbar(X, x, \alpha) = W_\hbar(X, x^g, \alpha), \quad \forall g \in \mathbb{R}^{\Delta_0(X)}.$$

The partition function (the case $\partial X = \emptyset$):

$$Z_\hbar(X, \alpha) = \int_{\mathbb{R}^{\Delta_1(X)}/\mathbb{R}^{\Delta_0(X)}} W_\hbar(X, x, \alpha) dx$$

Shape gauge transformations

Let X be a closed ($\partial X = \emptyset$) oriented triangulated pseudo 3-manifold where all tetrahedra are oriented, and all gluings respect the orientations with shape α.

Shape gauge group action in the space of shapes is generated by total dihedral angles around edges acting through the Neumann–Zagier Poisson bracket.

A gauge reduced shape is the Hamiltonian reduction of a shape over fixed values of the total dihedral angles.

An edge is balanced if the total dihedral angle around it is 2π. A shape with all edges balanced is known as an angle structure (Casson, Lackenby, Rivin).

Invariants of 3-manifolds

Theorem

For a closed oriented triangulated pseudo 3-manifold X with shape α, the partition function $Z_\hbar(X, \alpha)$ is well defined (the integral is absolutely convergent), and it

- *depends on only the gauge reduced class of α;*
- *is invariant under shaped $3-2$ Pachner moves along balanced edges.*

Remark

This construction can be extended to manifolds with boundary eventually giving rize to a TQFT.

One vertex H-triangulations of knots in 3-manifolds

Let $K \subset M$ be a knot in an oriented closed compact 3-manifold. Let X be a one vertex H-triangulation of the pair (M, K), i.e. a one vertex triangulation of M where K is represented by an edge e_0 of X.

Fix another edge e_1, and for any small $\epsilon > 0$, consider a shape structure α_ϵ such that the total dihedral angle is ϵ around e_0, $2\pi - \epsilon$ around e_1, and 2π around any other edge.

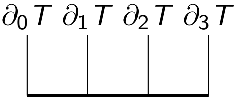
Theorem

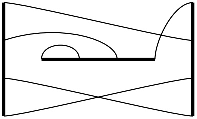
The limit

$$\tilde{Z}_\hbar(X) := \lim_{\epsilon \to 0} Z_\hbar(X, \alpha_\epsilon) \left| \Phi_\hbar \left(\frac{\pi - \epsilon}{2\pi i \sqrt{\hbar}} \right) \right|^2$$

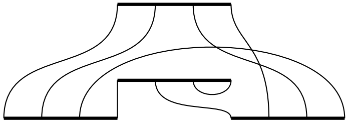
is finite and is invariant under shaped $3 - 2$ Pachner moves of triangulated pairs (M, K).

An H-triangulation of the pair $(S^3, 4_1)$ (figure-eight knot)

Graphical notation: T $=$ 



$$\tilde{Z}_\hbar(S^3, 4_1) = 2\left|\int_{\mathbb{R}-i\epsilon} \frac{e^{i\pi z^2}}{\Phi_\hbar(z)^2}\, dz\right|^2$$



$$\tilde{Z}_\hbar(S^3, 5_2) = 2\left|\int_{\mathbb{R}-i\epsilon} \frac{e^{i\pi z^2}}{\Phi_\hbar(z)^3} dz\right|^2$$

A conjectural relation to the Teichmüller TQFT

The Teichmüller TQFT (constructed in: J.E. Andersen–RK, arXiv:1109.6295)

Conjecture

For any closed 1-vertex triangulation of a closed 3-manifold X with shape α, one has

$$Z_\hbar(X,\alpha) = 2\left|Z_\hbar^{(\mathrm{Teichm.})}(X,\alpha)\right|^2$$